A local representation of Electromagnetism as a spacetime distortion versus a global representation as a Riemann Configuration Space

Abstract

It can be demonstrated that Relativistic corrections to equations of motion can be obtained without having to resort to the apparatus of Special Relativity and all its questionable notions.It is here conjectured that a guiding principle in the formulation of physical laws is that they should be independent of the details of the experimental apparatus employed in their verification. In this sense, the details of an experimental apparatus provide a more precise definition for what is traditionally referred to as a ‘frame of reference’. To elaborate, when one thinks of a frame of reference, images of measuring rods and clocks usually come to mind in order to record moments in time and positions in 3D space. In real experiments the hardware employed to gather data is usually much more sophisticated than measuring rods and clocks. The approach taken here is that each instrument reading in an experiment should be regarded as an independent dimension in a fundamental space. So in general the number of dimensions in the space can be many. The space of observations we will refer to as a configuration space and a frame of reference in this space will refer to the instrumentation used to make observations. Physical laws should not be based on simple abstractions, e.g. modeling particles as points moving in a Euclidean background space. Such simple abstractions as these have no basis in reality and exist only in the mind of the observer, they are merely shadowy projections of a deeper reality, at best useful approximations to be employed by engineers. If it is assumed that a physical law follows a Lagrangian description in all frames of reference, then it can be shown that a plausible form for the Lagrangian is that of a distance element in the configuration space embedded with a Riemann manifold which is referred to as a Riemann configuration space. The equations of motion generated by the distance element lagrangian are then simply the geodesics on the Riemann manifold. A first guess at the form of such a distance element lagrangian which can recover the classical lagrangian of classical

mechanics under normal conditions is conjectured to be,

$$L=\sqrt{\phi^2-2\ T\,\phi} \qquad (1)$$

ϕ is the total potential Energy and T is the total Newtonian kinetic energy.

In the limit where $T/\phi \to 0$, L can be approximated as

$$L=\phi-T$$

which is the classical Lagrangian of Classical Mechanics.

The limit $T/\phi \to 0$ is justified if one assumes that most of the potential energy is locked up in the mass of the system which for everyday systems is a huge quantity as compared to the interaction potential between different components (e.g. between particles) of the system and typical Newtonian/classical kinetic energies .

If one applies $L=\sqrt{\phi^2-2\ T\,\phi}$ to a system consisting of a single particle where $\phi=\phi_0$, the particle's internal energy, a constant, and $T=m\,\frac{u^2}{2}$ the particle's kinetic energy, then one obtains the system's energy to be

$$E=\frac{\phi_0}{\sqrt{\left(1-\frac{m\,u^2}{\phi_0}\right)}}$$

which is Einstein's mass –velocity relationship if $\phi_0=m\,c^2$

In other words the distance element lagrangian contains what is commonly known as the relativistic correction. With the Riemann configuration space formulation , in order to obtain the Relativistic corrections, it is not necessary to begin with the assumptions of Special Relativity or to deal with the paradox's inherent in its implications. What's more the distance element Lagrangian, naturally and easily allows for the analysis of a system of arbitrary size, not just a single point particle as in Special Relativity. However a limitation of equation (1) is that it is not invariant with respect to Galilean transformations. In what follows it will be shown how it is possible to construct a simple lagrangian which not only has the property of being invariant with respect to Galilean transformations but also any transformation which preserves spatial distance. Moreover such a lagrangian allows for the natural emergence of the electromagnetic and gravitational fields as well as simultaneously providing an explanation of inertial mass. It will be also shown that mass is in general a tensor quantity which only has the appearance of being a scaler quantity in an isentropic spherically symmetric universe, and that mass is an extrinsic property of a particle as opposed to the common intuition of being an intrinsic property.

The following analysis is a relational theory of mechanics and encapsulates all principles, as stipulated by Ernest Mach, of what such a theory should contain, primarily that such a theory should be fundamentally devoid of the need of a coordinate system and that any one observation only has meaning when compared to all other observations. It will also be demonstrated how Weber's force law and a generalization of Maxwell's field theory simultaneously emerge from the principles to be elucidated, and that the classical energy of a closed system is in general not a conserved quantity but which instead manifests itself as having the tendency to increase total universal entropy.

■ Riemann Manifolds and rotating coordinate frames

In physics the application of Riemann manifolds and the corresponding analytical apparatus is confined to namely the General Theory of Relativity and the light speed preserving Spacetime coordinate systems which place restrictions on the form of the metric, i.e. only transformations which preserve the speed of light are considered, and the spaces considered are 4D. 3 space and one time. In Non-relativistic theories i.e., classical or Newtonian mechanics, Riemann manifolds are confined to 3 space dimensions and the time is considered separate. The cited reason for this is that in general an arbitrary Spacetime transformation from one coordinate system to another will not preserve the speed of light, maybe so, but this should not prevent a non light speed preserving metric from expressing an equation of motion, for what is a geodesic in one coordinate system is also a geodesic in any other coordinate system, even those which do not preserve the speed of light. (In fact the requirement that the speed of light be preserved between transformations shall be ignored as it is not relevant to the formulation of mechanics presented in this paper) Realizing this key point allows the full power of Riemann metrics to be exploited. In relativity theory the metric is restricted so that the speed of light is always the same, and only transformations between such metrics are permissible. In this paper no such restrictions are placed on the metric, an arbitrary transformation from one reference system to another gives rise to a new metric in which perfectly reasonable equations of motion are obtained. For example consider two Cartesian coordinate systems on the plane. Let one be inertial and the other be rotating. By definition in the inertial frame free particles follow straight lines, and in the rotating frame the same particles follow curved lines. If it is assumed that the particles follow geodesics then it is easy to find a metric in the inertial system which give rise to the geodesics, i.e. choose

$G_{i,j}$ = Constant,

and it does not matter which constants are chosen, just so long as they are constant.

Let $G_{i,j}{}'$ denote the metric in the rotating system. $G_{i,j}{}'$ can easily be determined from $G_{i,j}$ if the transformation which takes the inertial points to the rotating frame is known. Once the $G_{i,j}{}'$ are determined, the paths can be calculated using the geodesic equations. For convenience choose

$$G_{i,j}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

then it is straight forward to calculate $G_{i,j}{}'$ to be

$$G_{i,j}{}'=\begin{pmatrix}1+w^2\,x_{2'}^2+w^2\,x_{3'}^2 & -w\,x_{3'} & w\,x_{2'}\\ -w\,x_{3'} & 1 & 0\\ w\,x_{2'} & 0 & 1\end{pmatrix}$$

$x_{2'} = x_2 \operatorname{Cos}[w\,x_1] - x_3 \operatorname{Sin}[w\,x_1]$
$x_{2'} = x_2 \operatorname{Sin}[w\,x_1] + x_3 \operatorname{Cos}[w\,x_1]$
$x_{1'} = x_1 = t$

Here w is the speed of rotation with respect to the inertial system.

Calculating the geodesics in the rotating system we get ;

$$\frac{dv^{1'}}{dt} = -2\ w^2\,v^{1'}\,(u^{2'}\ x_{2'} + u^{3'}\,x_{3'})$$

now $\frac{dv^{1'}}{dt}=\frac{dv^1}{dt}=0$ because $v^1=\frac{dt}{ds}$ which in the inertial frame is constant, which implies that

$(u_{2'}\,x_{2'} + u_{3'}\,x_{3'}) = 0$

For the Geodesics corresponding to the spatial coordinates we have

$$\frac{du^{2'}}{dt} = -w^2\ (x_2 + 2\,w\,x_3\,(u^{2'}\ x_{2'} + u^{3'}\,x_{3'})) = -w^2\,x_2$$

$$\frac{du^{3'}}{dt} = -w^2 \ (x_3 + 2\, w\, x_3 \,(u^{2'}\ x_{2'} + u^{3'}\, x_{3'}))=-w^2\, x_3$$

This simple example serves to illustrate the point that the metric space apparatus and space-time transformations can be employed in a description of non relativistic transformations
or transformations which are not Lorentz invariant.

■ What is Spacetime?

Spacetime having four dimensions is an abstract model with no verifiable basis in reality because it lacks concrete definition. All that exists are measurements and the relations between them, which are what physical laws are. Each measurement or observable is a dimension in its own right. For example if an experiment is being performed in which ten observables are being recorded then the space is ten dimensional or eleven dimensional if a clock is being employed as one of the instruments. Also if a clock is being used, then using more than one would add redundancy as well as present calibration problems. After performing the experiment, it may be possible to formulate a summary of the collected data or to deduce a law which can be used to predict the outcome of future experiments. If such a predictive law is formulated then it only has meaning in the context defined by the details of the experimental apparatus or the 'frame of reference' defined by the apparatus used to gather the data. Any predictions which can be made can only be verified using the same experimental setup or within the same 'frame of reference'. Suppose two experiments are been concurrently performed, collecting the same data but using different Instrumentation.

y1(x1......x10)
y2(x1......x10)
.....
y10(x1.....x10)

The x's are the readings on the first experimenter's instruments and the y's are the readings on the second experimenter’s set of instruments. Also note that once a physical law is deduced from an examination of the experimental data,it alone is not enough to describe the phenomenon. What is also required is a description of the instrumentation used to collect the data. This way if the same phenomenon is being observed using different instrumentation, inferences can be made by transforming the readings from one set of instruments to the other. As an example Suppose an experiment is being performed to measure the interaction between two particles. Seven instruments it is felt should supply enough information to determine a summary of the interaction. Six transducers to measure the x, y and z coordinates of the particles, if a rectangular reference system is chosen and one clock. Note, an entire array of clocks placed at different points in the coordinate system, it is felt, is not required, as this would supply to much information as well as lead to calibration problems. So the system can be thought of as existing in a seven dimensional space, it may be also be possible to obtain a predictive summary by only having two instruments, one, a transducer to measure the particle separation and a clock, in which case the problem is only two dimensional. In any case each configuration is regarded as a distinct point in the 7D or 2D space, which ever is being attempted, and as the measurements evolve they trace out a path in the space. One approach would be to assume that the permitted paths follow geodesics in the configuration space. To do this requires imposing a metric on the configuration space to form a Riemann manifold where neighboring points are related by

$\Delta x_i \, \Delta x_j \, g_{ij} = ds^2$, $i = 1, 7$, $j = 0, 6$, where x_1 is the reading on the clock.

If the metric g_{ij} can be determined then it is straightforward to calculate the path by using the geodesic equations. Moreover if the same phenomenon is observed using different instrumentation, i.e. in a different configuration space, x'_i ,and the transformation is known, i.e. $x'_i \, (x_i)$, then the g_{ij}' can be easily determined and hence geodesics calculated. Imposing a metric on the configuration space does not imply that the physical background space (PBS) is curved. The PBS does not really exist, only measurements exist and the relationships between them. In fact we are not placing any restriction on the kinds of experiments being performed and the observables may not even have anything to do with the so called PBS. For example each of the observables could be commodity prices just to make the point, but for the most part, the spirit of this investigation shall be directed at physical phenomenon. In what follows the observables is assumed to have a Lagrangian description. By considering a certain class of lagrangians it will be strongly suggested that geodesic motion in configuration spaces is the dominant one for a wide range of natural phenomenon, not just gravity.\ n The concept of a configuration space is not new, and in fact treating all observables as a single point in a multi dimensional space is implicit in the Lagrangian formulation of mechanics. However the idea of embedding a Riemann metric on such a multi dimensional space would appear to be a novel one and one which is a focal point of the analysis presented in this paper. Traditionally when Riemann manifolds have been employed in mechanics, as in General Relativity (GR), the approach has been to model the background 4 space-time as a Riemann manifold on which particles move and from which observations are taken. In GR the properties of the manifold are determined from the distribution of the particles, or more generally the distribution of matter and energy, and each particle is assumed to move along geodesic paths in the 4D space-time manifold. If a particle does not follow a geodesic, as for example an electron in an electric field, then the use of a 4D space-time manifold fails to provide a complete description, hence the need for a unified field theory. Approaches to finding such a theory have centered on extending the number of dimensions that make up the the physical background space and reexpressing

Einstein's equations of General Relativity in these higher dimensional spaces. Such approaches have met with varying degrees of success and skepticism alike and have their current incarnation in String Theory and M Theory. The extra 'physical' dimensions required to bring about unification are curled up in little hyper spheres and are therefore hidden from everyday observation and participation in experiments.

By regarding all experimental observations as a single point in a multi dimensional Riemann manifold, the interactions and couplings between each of the observables arises naturally by virtue of the metric coefficients. In the analysis to follow it will be shown that the most natural form (and it is conjectured but not proven to be the only form) the system's lagrangian can take is that of the distance element in the multidimensional metric space, and therefore the system evolves so as to trace out the shortest distance in the Riemann Configuration Space. The metric coefficients are determined by requiring the Lagrangian reproduce Einstein's mass-energy relationship for a single particle and that the classical lagrangian (T-V) is recovered when all velocities are small when compared to the speed of light. The 'Distance Element' Lagrangian, in essence, represents what has been traditionally termed, the Relativistic correction, and it will be shown how it consistently and naturally extends beyond the single particle case.

■ The Distance Element Lagrangian

Consider an experiment with n observables x_i and a clock, t and that the evolution of the observables have a Lagrangian description, L_u, the form of which at this point is not known. Also form a metric space by imposing the metric g_{ij} on the configuration space $(x_i, x_0 = t, i = 1, n)$

$$L_u\,[u^i, x_i, t]\,,\ i = 1, n\ ,\ u^i = \frac{dx_i}{dt}$$

$$\text{Satisfies } \frac{d}{dt}\left[\frac{\delta L_u}{\delta u^i}\right] - \frac{\delta L_u}{\delta x_i} = 0\,,\ i = 1, n. \qquad (1.1)$$

The u^i quantities are the rates of change of each observable with respect to the reading on the particular clock used in the experiment

$$\text{define } \Delta\phi = \Delta\phi[\Delta x_i, \Delta t]\,,\ v^i = \frac{\Delta x_i}{\Delta\phi}\,,\ v^0 = \frac{\Delta t}{\Delta\phi}$$

$$= >\ v^i = u^i\, v^0\,,\ i = 1, n.$$

Here $\Delta\phi$ is a scalar quantity formed by changes in all observables. In this sense, ϕ can be regarded as a universal time variable, the same in all 'Reference Frames' and independent of the particular details of an experimental setup

The v^i quantities are simply rates of change with respect to universal time and are Covariant vectors with respect to the Riemann Configuration Space, g_{ij}

define

$$L_v[v^j, x_j] = L_u[u^i, x_i, t]\,,\ i = 1, n,\ j = 0, n\ , x_0 = t.$$

L_v always exists since

$$L_u[u^i, x_i, t] = L_u\left[\frac{v^i}{v_0}, x_i, t\right],\ i = 1, n$$

which = >

$$\Delta L_u = \frac{\delta L_u}{\delta u^i}\,\Delta u^i + \frac{\delta L_u}{\delta x_i}\,\Delta x_i + \frac{\delta L_u}{\delta x_0}\,\Delta x_0 = \Delta L_v = \frac{\delta L_v}{\delta v^i}\,\Delta v^i + \frac{\delta L_v}{\delta v^0}\,\Delta v^0 + \frac{\delta L_v}{\delta x_i}\,\Delta x_i + \frac{\delta L_v}{\delta x_0}\,\Delta x_0$$

Now since

$$\Delta v^i = v^0\,\Delta u^i + \Delta v^0\,u^i$$

then

$$\Delta L_v = v^0\,\frac{\delta L_v}{\delta v^i}\,\Delta u^i + \left[u^k\,\frac{\delta L_v}{\delta v^k} + \frac{\delta L_v}{\delta v^0}\right]\Delta v^0 + \frac{\delta L_v}{\delta x_i}\,\Delta x_i \quad \text{where } i = 1, n \text{ and } k = 1, n\,.$$

In general

$$v0 = v0\,(u_i, x_i, x_0)\,,$$

so equating the coefficients of the Δu^i's and Δx_i'
s is NOT IMPLIED here. But in what follows the classes of L_v considered will be such that,

$$u^k\,\frac{\delta L_v}{\delta v^k} + \frac{\delta L_v}{\delta v^0} = 0\quad k = 1, n$$

$$\text{or}\ \ v^k\,\frac{\delta L_v}{\delta v_k} = 0\ \text{with } k = 0, n \qquad (1.2)$$

which means that,

$$v^0 \frac{\delta L_v}{\delta v^i} = \frac{\delta L_u}{\delta u^i}, \ i = 1, n \qquad (1.3)$$

and that ,

$$\frac{\delta L_v}{\delta x_i} = \frac{\delta L_u}{\delta x_i} \ i = 0, n\,. \qquad (1.4)$$

Combining equations (1.1) , (1.3) and (1.4) yields

$$\frac{d}{d\phi}\left[\frac{\delta L}{\delta v^i}\right] - \frac{\delta L}{\delta x_i} = 0 \quad i = 1,$$

n \ (1.5)

where $L = v^0 L_v$.

For the case in which $i = 0$

$$\frac{d}{d\phi}\left[\frac{\delta L}{\delta v^0}\right] - \frac{\delta L}{\delta x_0} = v^0\left[\frac{d}{dt}\left(L_u - u^i \frac{\delta L_u}{\delta u^i}\right) - \frac{\delta L_u}{\delta t}\right], \quad \text{(using equations (1.2) , (1.3) and (1.4))}$$

$$= v^0\left[\frac{\delta L_u}{\delta t} + \frac{\delta L_u}{\delta x_i} u^i + \frac{\delta L_u}{\delta u^i}\frac{du^i}{dt} - \frac{\delta L_u}{\delta u^i}\frac{du^i}{dt} - \frac{\delta L_u}{\delta x_i} u^i - \frac{\delta L_u}{\delta t}\right] = 0 \qquad \text{(using equation (1.1)).}$$

Therefore

$$\frac{d}{d\phi}\left[\frac{\delta L}{\delta v^i}\right] - \frac{\delta L}{\delta x_i} = 0 \qquad \text{for } i = 0, n \qquad (1.6)$$

The case for $i = 0$ describes the energy of the system.

A Hamiltonian for the system (1.6) can be defined as

$$H = v^i \frac{\delta L}{\delta v^i} - L$$

which is not the same for the case when $i = 0$ in equation (1.6). H can be thought of as a hyper energy of the system . From equation (2) it can be easily seen that

$$H = v^i \frac{\delta L}{\delta v^i} - L \quad = 0 \qquad (1.7)$$

H is a scalar and must be identically zero in arbitrary frames of reference, since L is a scalar and v^i and $\frac{\delta L}{\delta v^i}$ transform as contravariant and covariant vectors respectively Equation (7) places a restriction on the form of L, namely that

$$L = \sum_{n=1}^{\infty} \gamma_n L[n]$$

$$L[n] = (G_{i_1,i_2 \ldots\ldots i_n} v^{i_1} v^{i_2} \ldots\ldots v^{i_{n-1}} v^{i_n})^{\frac{1}{n}} \qquad (1.8)$$

is a solution , and is here conjectured to be the most general form which satisfies eq (1.7) .

An example of L could be

$$L = A \sqrt{g_{ij} v^i v^j} + B \frac{\delta \Psi}{\delta x_i} v^i$$

in which g_{ij} is the metric tensor and Ψ is a scalar potential and A and B are constants.

The special solution

$$L = L[2] = (G_{i,j} v^i v^j)^{\frac{1}{2}}$$

we refer to as the distance element Lagrangian
Null

Determining $\Delta\phi$

$$\frac{d}{d\phi}\left[\frac{\delta L}{\delta v^i}\right] - \frac{\delta L}{\delta x_i} = 0 \qquad \text{for } i = 0, n.$$

Since L is a scalar function then

$$L[x_i, v^i] = L'[x'_i, v'^{\,i}].$$

However it is not clear that ,

$$\frac{d}{d\phi}\left[\frac{\delta L'}{\delta v'^{\,i}}\right] - \frac{\delta L'}{\delta x_i'} = 0$$

In what follows this will be shown to be in fact the case.

Suppose (x_i, g_{ij}) forms a metric space , M , so that $\Delta x^i \Delta x^j g_{ij} = ds^2$.

Define the basis vectors e_i such that $e_i \Delta x^i = e'_i \Delta x'^i$, $e^i = g^{i,j} e_j$ where $g^{i,j}$ is the inverse of g_{ij} .

It can be shown that,

$$(v^i \Gamma_{i,j}{}^k \Delta x^j + \Delta v^k) e_k \qquad (1.9)$$

transforms as a vector .

Also $$\Delta L = \frac{\delta L}{\delta v^i} \Delta v^i + \frac{\delta L}{\delta x^i} \Delta x^i \qquad (1.10)$$

is a scalar .

When $\Delta x^i = 0$, (1.9) = > that $\Delta v^k e_k$ transforms as a vector

which = > that from (1.10) that

$\frac{\delta L}{\delta v^i}$ must transform as a vector also, which in turn = > that

$$\frac{\delta L}{\delta v^k} (v^i \Gamma_{i,j}{}^k \Delta x^j + \Delta v^k) \qquad (1.11)$$

must transform as a scalar for non zero Δx^j.

Subtracting (1.11) and (1.10) gives

$$\left(\frac{\delta L}{\delta x^j} - \frac{\delta L}{\delta v^i} v^i \Gamma_{i,j}{}^k \right) \Delta x^j ,$$

which transforms as a as scalar which

= > that

$$\left(\frac{\delta L}{\delta x^j} - \frac{\delta L}{\delta v^k} v^i \Gamma_{i,j}{}^k \right) e^j$$ will transform as a vector.

Define $P = \frac{\delta L}{\delta v^j} e^j$, the momentum vector, and

$$\text{grad}(L) = \left(\frac{\delta L}{\delta x^j} - \frac{\delta L}{\delta v^k} v^i \Gamma_{i,j}{}^k \right) e^j.$$

Then $\frac{dP}{d\phi} - \text{grad}(L)$ is the same in all coordinate systems and further more

$$\frac{dP}{d\phi} - \text{grad}(L) = \left[\frac{d}{d\phi} \left(\frac{\delta L}{\delta v^j} \right) - \frac{\delta L}{\delta x^j} \right] e^j = 0 \qquad (1.12)$$

which = > that

$\frac{d}{d\phi} \left(\frac{\delta L}{\delta v^j} \right) - \frac{\delta L}{\delta x^j} = 0$ is true in all coordinate systems since the e^j are linearly independent

and $\frac{\delta L}{\delta v^j} \frac{de^j}{d\phi} = - \frac{\delta L}{\delta v^k} v^i \Gamma_{i,j}{}^k e^j$.

Now since L and $d\phi$ are scalers , the the action quantity

$$A = \int_{p1}^{p2} L \, d\phi$$

is a scalar across all coordinate systems , ie

$A' = A$.

Let us restrict ourselves to the case when the action is stationary , ie when equation (1.6) is true.

Let R' denote the primed coordinate system and R the unprimed coordinate system

let $T[\beta^i]$ represent transformations from $R' \to R$,
ie $T[\beta^i] : R' \to R$, where the β^i are independent parameters
which can be varied at will to produce different coordinates systems R.

Suppose for $\beta^i = 0$, $x'^i \to x^i$ and $v'^i \to v^i$ is given by T[0],

i.e

$$A[\beta^i = 0] = \int_{p1}^{p2} L[\beta^i = 0, x^i, v^i]\, d\phi = \int_{p1}^{p2} L'[x'^i, v'^i]\, d\phi' = A' \qquad (1.13)$$

and that

$$\frac{d}{d\phi}\Big[\frac{\delta L}{\delta v^i}\Big] - \frac{\delta L}{\delta x_i} = 0\ ,\ \frac{d}{d\phi}\Big[\frac{\delta L'}{\delta v'^i}\Big] - \frac{\delta L'}{\delta x_i'} = 0.$$

Now let the transformation be perturbed by shifting the $\beta^i = 0$
by $\Delta\beta^i$ so that

$x'^i \to x^i + \Delta x^i$

and

$v'^i \to v^i + \Delta v^i$

are given by $T[\Delta\beta^i]$. Under the new transformation the Action

$A[\Delta\beta^i] = A[0]$

since all we are doing is changing the reference
system R and scalar quantities don't care about the details
of the reference system . Like wise

$d\phi[0] = d\phi[\Delta\beta^i]$

All taken together implies that

$$0 = \Big[\frac{\delta L}{\delta v^i}\frac{\delta x^i}{\delta \beta^i}\Big]_{p1}^{p2} + \int_{p1}^{p2} \frac{d}{d\phi}\Big[\frac{\delta L}{\delta v^i}[\beta^i = 0]\Big] - \frac{\delta L}{\delta x_i}[\beta^i = 0]\, d\phi = \Big[\frac{\delta L}{\delta v^i}\frac{\delta x^i}{\delta \beta^i}\Big]_{p1}^{p2} \qquad (1.14)$$

Now the approach taken here is that which is typically
employed to prove Noether's theorem to show conservation laws.
However unlike Noether's theorem we are not restricting
ourselves to any specific reference system e.g. cartesian , or a
specific class of transformations, i.e. translational or rotational . No
restrictions are being placed on the reference system , R,

or the transformations, $T[\beta^i]$.

Consequently since the $\frac{\delta x^i}{\delta \beta^i}$ are arbitrary, the only way for (1.14) to be true is for

$$\frac{\delta L}{\delta v^i} = 0 \ ,$$

which = > that from (1.6) that

$$\frac{\delta L}{\delta x_i} = 0$$

which in turn

= > L = K,
a constant
(1.15)

How can (1.15) be reconciled with (1.8). The solution is in the choice of $\Delta\phi$.

If

$$L = \sum_{n=1}^{\infty} \gamma_n L[n]$$

$$L[n] = (G_{i_1,i_2 \ldots\ldots i_n} v^{i_1} v^{i_2} \ldots\ldots v^{i_{n-1}} v^{i_n})^{\frac{1}{n}} \ ,$$

then choose

$$\Delta\phi[n] = (G_{i_1,i_2 \ldots\ldots i_n} dx^{i_1} dx^{i_2} \ldots\ldots dx^{i_{n-1}} dx^{i_n})^{\frac{1}{n}} .$$

This means that

$$K \Delta\phi = \sum_{n=1}^{\infty} \gamma_n \Delta\phi[n]$$

which = > that

$L = K$

This gives the form of L_u to be

$$L_u = \frac{K}{v_0} = K \frac{\Delta\phi}{\Delta t} = \sum_{n=1}^{\infty} \gamma_n \frac{\Delta\phi[n]}{\Delta t} = \sum_{n=1}^{\infty} \gamma_n \left(G_{i_1,i_2 \ldots\ldots i_n} u^{i_1} u^{i_2} \ldots \ldots u^{i_{n-1}} u^{i_n}\right)^{\frac{1}{n}}$$

where $u^0 =$
1.
(1.16)

Going back to our earlier example but in this case for L_u , gives;

$$L_u = A \sqrt{g_{ij} u^i u^j} + B \frac{\delta\Psi}{\delta x_i} u^i$$

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Null
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Conservation laws

Conservation laws can be determined from an
inspection of the lagrangian alone and the approach employed
in Noether's theorem is not required. For example suppose that in a particular reference system the
lagrangian, L, is found to have the property that

$$L[x_0, \ldots .., x_q, \ldots x_p .. x_n] = L[x_0, \ldots .., x_q + \lambda, \ldots x_p + \lambda .. x_n]$$

then this implies that

$$\frac{\delta L}{\delta x_q} + \frac{\delta L}{\delta x_q} = 0,$$

which from (1.6) gives

$$\frac{d}{d\phi}\left[\frac{\delta L}{\delta v^q} + \frac{\delta L}{\delta v^p}\right] = 0,$$

.i.e. a conservation of momentum type law.

What's more,
conservation laws are dependent on the frame of reference in which the lagrangian is inspected

Conservation of energy arises if the L in the above
example is independent of time or if there is a translational invariance
with respect to x_0, i.e. if

$$L[x_0, \ldots \ldots x_n] = L[x_0 + \lambda, \ldots \ldots x_n]$$

which again from (1.6) gives ,

$$\frac{d}{dt}[E]\frac{dt}{d\phi} = 0$$

or simply,

$$\frac{d}{dt}[E] = 0\,,$$

where $E = \dfrac{\delta L}{\delta v^0}$. (1.17)

Again let

$$L = A\sqrt{g_{ij}\, v^i\, v^j} + B\frac{\delta \Psi}{\delta x_i} v^i\,.$$

In this case E becomes,

$$E = A\frac{g_{0\,i}\, v^i}{\sqrt{g_{ij}\, v^i\, v^j}} + B\frac{\delta \Psi}{\delta x^0} =$$

$$A\frac{g_{0\,i}\, u^i}{\sqrt{g_{ij}\, u^i\, u^j}} = A v_0 \tag{1.18}$$

C oming back to the conservation of momentum example with the example lagrangian

$$\frac{d}{d\phi}\left[\frac{\delta L}{\delta v^q} + \frac{\delta L}{\delta v^p}\right] =$$

$$A\frac{d}{d\phi}[v_q + v_p] = 0 \tag{1.19}$$

Newtonian correspondence

Allow a system consisting of n observables to have to have a total potential energy , $\Phi(x_1, \,..\, x_n, t)$ and a total Newtonian kinetic energy, $T(u^1 \,....u^n)$. The Classical Lagrangian, L_c, has the form;

$$L_c = \Phi - T \qquad (2.1)$$

For L_u to be in agreement with L_c when the system's velocities are within the domain of everyday observations, the following form of L_u is chosen,

$$L_u = \sqrt{\Phi^2 - 2\,\Phi\,T} \qquad (2.2)$$

$$L_u \approx L_c \quad \text{if it is assumed that } \frac{T}{\Phi} \approx 0 \quad \text{since } L_u = \Phi\left(1 - \frac{2\,T}{\Phi}\right)^{\frac{1}{2}} \approx \Phi - T = L_c \qquad (2.3)$$

Using L_u in Eq (1.18) the energy, E, of the system is obtained as,

$$E = \frac{\Phi}{(1 - \frac{2\,T}{\Phi})^{\frac{1}{2}}} \qquad (2.4)$$

For a free particle of rest mass m and moving with velocity u,

$$T = \frac{1}{2}\,m\,u^2 \qquad (2.5)$$

If we now assume that the potential energy Φ corresponds to the rest mass energy of the particle, ie

$$\Phi = m\,c^2 \qquad (2.6)$$

then

$$E = \frac{m\,c^2}{\left(1 - \frac{u^2}{c^2}\right)^{\frac{1}{2}}} \qquad (2.8)$$

which is the Einstein Energy – velocity relationship.

Correction to the Newtonian equations of motion

Define

$$l_c\,(i) = \frac{d}{dt}\,\frac{\partial}{\partial u^i} - \frac{\partial}{\partial x_i} \qquad (3.1)$$

as the Lagrangian operator.

$l_c\,(i)[L_c] = 0$ (3.2)

defines the non Relativistic equations of motion.

However because of the form of L_u in eq (2.2) , eq (3.2) should receive a modification of the form;

$l_c\,(i)\,L_u = F_i$ (3.3)

where F_i should be small under everyday conditions . In order to quantify F_i the following approach is taken .

Observe that

$L_c{}^2 = L_u{}^2 + T^2$ (3.4)

Since $l_c\,(i)$ is a linear operator then;

$l_c\,(i)\,L_c{}^2 = l_c\,(i)\,L_u{}^2 + l_c\,(i)\,T^2$ (3.5)

also for any functional L,

$$l_c\,(i)\,L^2 = 2\,L\,l_c\,(i)\,L + \frac{\partial L}{\partial u^i}\,\frac{dL}{dt} \qquad (3.6)$$

Therefore since $l_c\,(i)\;L_u = 0$

$$\frac{\partial L_u}{\partial u^i}\,\frac{dL_u}{dt} = L_c\,l_c\,(i)\,L_c + \frac{\partial L_c}{\partial u^i}\,\frac{dL_c}{dt} - \left(T\,l_c\,(i)\,T + \frac{\partial T}{\partial u^i}\,\frac{dT}{dt}\right) \qquad (3.7)$$

which is the same as

$$L_c\,l_c\,(i)\,L_c = \frac{\partial L_u}{\partial u^i}\,\frac{dL_u}{dt} - \frac{\partial L_c}{\partial u^i}\,\frac{dL_c}{dt} + T\,l_c\,(i)\,T + \frac{\partial T}{\partial u^i}\,\frac{dT}{dt} \qquad (3.8)$$

Using the fact that

$T = L_c + \Phi$

and that

$$l_c\,(i)\,\Phi = -\frac{\partial \Phi}{\partial x_i}$$

means that eq (3.8) simplifies to

$$(T - \Phi)\, l_c\,(i)\, L_c = \frac{\partial L_u}{\partial u^i}\frac{dL_u}{dt} - \frac{\partial L_c}{\partial u^i}\frac{dL_c}{dt} + T\, l_c\,(i)\,(L_c + \Phi) + \frac{\partial T}{\partial u^i}\frac{dT}{dt} \qquad (3.9)$$

or

$$\Phi\, l_c\,(i)\, L_c = -\frac{\partial L_u}{\partial u^i}\frac{dL_u}{dt} + \frac{\partial L_c}{\partial u^i}\frac{dL_c}{dt} + T\frac{\partial \Phi}{\partial x_i} - \frac{\partial T}{\partial u^i}\frac{dT}{dt} = -\frac{\partial L_u}{\partial u^i}\frac{dL_u}{dt} - \frac{\partial T}{\partial u^i}\frac{dT}{dt} + \frac{\partial T}{\partial u^i}\left(\frac{dT}{dt} - \frac{d\Phi}{dt}\right) + T\frac{\partial \Phi}{\partial x_i}$$

$$= -\frac{\partial L_u}{\partial u^i}\frac{dL_u}{dt} - \frac{\partial T}{\partial u^i}\frac{dT}{dt} + \frac{\partial T}{\partial u^i}\left(\frac{dT}{dt} - \frac{d\Phi}{dt}\right) + T\frac{\partial \Phi}{\partial x_i}$$

$$= -\frac{\partial L_u}{\partial u^i}\frac{dL_u}{dt} - \frac{\partial T}{\partial u^i}\frac{d\Phi}{dt} + T\frac{\partial \Phi}{\partial x_i} \qquad (3.10)$$

(3.10) can be simplified further by exploiting the relationships

$$E = \frac{\Phi^2}{L_u} \quad \text{and} \quad \frac{dL_u}{dt} = \frac{2\,\Phi}{E}\frac{d\Phi}{dt} \quad \text{(as E is a constant of the motion)}$$

$$\frac{\partial L_u}{\partial u^i} = -\frac{\Phi}{L_u}\frac{\partial T}{\partial u^i} = -\frac{E}{\Phi}\frac{\partial T}{\partial u^i}$$

$$T = \frac{\Phi}{2}\left(1 - \frac{\Phi^2}{E^2}\right)$$

to give;

$$l_c\,(i)\, L_c = \frac{\partial T}{\partial u^i}\frac{\frac{d\Phi}{dt}}{\Phi} + \frac{T}{\Phi}\frac{\partial \Phi}{\partial x_i} = \frac{1}{2}\left(1 - \frac{\Phi^2}{E^2}\right)\frac{\partial \Phi}{\partial x_i} + \frac{\partial T}{\partial u^i}\frac{\frac{d\Phi}{dt}}{\Phi} \qquad (3.11)$$

which implies that

$$F_i = \frac{1}{2}\left(1 - \frac{\Phi^2}{E^2}\right)\frac{\partial \Phi}{\partial x_i} + \frac{\partial T}{\partial u^i}\frac{\frac{d\Phi}{dt}}{\Phi} \qquad (3.12)$$

Under everyday conditions $\frac{\Phi^2}{E^2} \approx 1$ and $\frac{\frac{d\Phi}{dt}}{\Phi} \approx 0$

since most of Φ is tied up in the constant rest mass of the system which means that

$$F_i \approx 0$$

Since F_i is non conservative then the change to the Newtonian energy defined by,

$$E_c = \sum_{i=1}^{n} u^i \frac{\partial L_c}{\partial u^i} - L_c \qquad (3.13)$$

can be computed to give

$$\frac{dE_c}{dt} = \sum_{i=1}^{n} u^i F_i = \frac{\frac{d\Phi}{dt}}{\Phi} \left(T + \sum_{i=1}^{n} u^i \frac{\partial T}{\partial u^i} \right) \qquad (3.14)$$

for the case when

$$\sum_{i=1}^{n} u^i \frac{\partial T}{\partial u^i} = 2\,T$$

eq (3.14) becomes

$$\frac{dE_c}{dt} = \frac{3\,T \frac{d\Phi}{dt}}{\Phi} = \frac{3}{2} \left(1 - \frac{\Phi^2}{E^2} \right) \frac{d\Phi}{dt} \qquad (3.15)$$

Galilean Invariance

The above analysis can be repeated for a modification of the lagrangian L_u which makes it invariant with regard to Galilean transformations.

More precisely consider a one dimensional problem, and L_u to be of the form ;

$$L_u = \sqrt{\Phi^2 - 2\,\Phi\,T + c^2\,P^2} \qquad (3.16)$$

where

$$P = \sum_{i=1}^{n} \frac{\partial T}{\partial u^i} \qquad (3.17)$$

the total Newtonian momentum .

The above analysis can be repeated for the modified form of L_u to give

$$l_c\,(i)\,L_c = \frac{-c^2}{\Phi} \frac{\partial P}{\partial u^i} \left(\frac{2\,P}{\Phi} \frac{d\Phi}{dt} + \frac{dP}{dt} \right) + F_i = F_i' \qquad (3.18)$$

which ignores the $l_c\,(i)\,P$ term by assuming that

$l_c\,(i)\,P \approx 0.$

Observe that

$$\sum_{i=1}^{n} \frac{\partial L_u}{\partial u^i} = \sum_{i=1}^{n} \frac{1}{L_u}\left(c^2\,P\,\frac{\partial P}{\partial u^i} - \Phi\,\frac{\partial T}{\partial u^i}\right) = \frac{P}{L_u}\left(c^2\sum_{i=1}^{n}\frac{\partial P}{\partial u^i} - \Phi\right) \qquad (3.19)$$

It is reasonable to assume that

$$c^2\sum_{i=1}^{n}\frac{\partial P}{\partial u^i} - \Phi = 0 \qquad \backslash \qquad (3.20)$$

which therefore implies that

$$\sum_{i=1}^{n} \frac{\partial L_u}{\partial u^i} = 0 \qquad (3.21)$$

which in turn from the Lagrangian equations yields

$$\sum_{i=1}^{n} \frac{\partial L_u}{\partial x_i} = 0$$

or that

$$\sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i} = 0 \qquad (3.22)$$

ie L_u, and consequently Φ, necessarily have the property that they have translational invariance, i.e for arbitrary λ,

$$L_u\,[x_i + \lambda,\, u^i] = L_u\,[x_i,\, u^i]\ ,\ i = 1,\ n. \qquad (3.23)$$

To calculate the magnitude of the term, $\frac{-c^2}{\Phi}\,\frac{\partial P}{\partial u^i}\left(\frac{2\,P}{\Phi}\,\frac{d\Phi}{dt} + \frac{dP}{dt}\right)$, in eq (3.18) it is necessary to explicitly calculate $\frac{dP}{dt}$ in terms of P and Φ. This can be achieved by noting that

$$\frac{dP}{dt} = \sum_{i=1}^{n} l_c\,(i)\,L_c = \sum_{i=1}^{n} \frac{-c^2}{\Phi}\,\frac{\partial P}{\partial u^i}\left(\frac{2\,P}{\Phi}\,\frac{d\Phi}{dt} + \frac{dP}{dt}\right) + \sum_{i=1}^{n}\left(\frac{1}{2}\left(1 - \frac{\Phi^2}{E^2}\right)\frac{\partial \Phi}{\partial x_i} + \frac{\partial T}{\partial u^i}\,\frac{\frac{d\Phi}{dt}}{\Phi}\right) =$$

$$\left(\frac{2\,P}{\Phi}\,\frac{d\Phi}{dt}+\frac{dP}{dt}\right)\sum_{i=1}^{n}\frac{-c^2}{\Phi}\,\frac{\partial P}{\partial u^i}+\frac{1}{2}\left(1-\frac{\Phi^2}{E^2}\right)\sum_{i=1}^{n}\frac{\partial \Phi}{\partial x_i}+\frac{\frac{d\Phi}{dt}}{\Phi}\sum_{i=1}^{n}\frac{\partial T}{\partial u^i}$$

$$=-\left(\frac{2\,P}{\Phi}\,\frac{d\Phi}{dt}+\frac{dP}{dt}\right)+\frac{\frac{d\Phi}{dt}}{\Phi}\,P \qquad (3.24)$$

which can be simplified to read

$$\frac{dP}{dt} \;=\; = -\frac{P}{2\,\Phi}\,\frac{d\Phi}{dt} \qquad (3.25)$$

Therefore

$$\frac{-c^2}{\Phi}\,\frac{\partial P}{\partial u^i}\left(\frac{2\,P}{\Phi}\,\frac{d\Phi}{dt}+\frac{dP}{dt}\right)=\frac{-c^2}{\Phi}\,\frac{\partial P}{\partial u^i}\left(\frac{2\,P}{\Phi}\,\frac{d\Phi}{dt}-\frac{P}{2\,\Phi}\,\frac{d\Phi}{dt}\right)$$

$$=\frac{-c^2}{\Phi^2}\,\frac{\partial P}{\partial u^i}\,\frac{3\,P}{2}\,\frac{d\Phi}{dt}\approx 0 \qquad (3.26)$$

under everyday conditions since

$$\frac{c^2}{\Phi^2}\approx\frac{1}{M^2\,c^2}\approx 0$$

where M is the total mass of the system . Therefore the inclusion of the $c^2\,P^2$ term does not cause any significant departure from the Newtonian approximation To demonstrate Galilean invariance of

$$L_u = \sqrt{\Phi^2-2\,\Phi\,T+c^2\,P^2}\;,$$

an ensemble of particles is considered where each of the particles is confined to move in one dimension and it is assumed that the total potential Φ is the sum of the rest mass energies,
i.e. ,

$$\Phi = c^2\sum_{i=1}^{n} m_i \qquad (3.27)$$

Substituting the Newtonian definitions for T and P , L_u becomes,

$${L_u}^2 = \Phi^2-2\,\Phi\,T+c^2\,P^2 = c^4\left(\sum_{i=1}^{n} m_i\right)^2-2\,c^2\left(\sum_{i=1}^{n} m_i\right)\left(\sum_{i=1}^{n}\frac{1}{2}\,m_i\,{u^i}^2\right)+c^2\left(\sum_{i=1}^{n} m_i\,u^i\right)^2 =$$

$$c^4\sum_{i,j=1}^{n} m_j\,m_i - c^2\sum_{i,j=1}^{n} m_i\,m_j\,{u^j}^2 + c^2\sum_{i,j=1}^{n} m_i\,m_j\,u^i\,u^j \;=$$

$$c^4 \sum_{i,j=1}^{n} m_j\, m_i - c^2\, \frac{1}{2} \left(\sum_{i,j=1}^{n} m_i\, m_j\, {u^j}^2 + \sum_{i,j=1}^{n} m_i\, m_j\, {u^i}^2 \right) + c^2 \sum_{i,j=1}^{n} m_i\, m_j\, u^i\, u^j =$$

$$c^4 \sum_{i,j=1}^{n} m_j\, m_i - c^2\, \frac{1}{2} \left(\sum_{i,j=1}^{n} m_i\, m_j\, {u^j}^2 + \sum_{i,j=1}^{n} m_i\, m_j\, {u^i}^2 - 2 \sum_{i,j=1}^{n} m_i\, m_j\, u^i\, u^j \right) =$$

$$c^4 \sum_{i,j=1}^{n} m_j\, m_i - c^2\, \frac{1}{2} \left(\sum_{i,j=1}^{n} m_i\, m_j\, (u^j - u^i)^2 \right)$$

$$= c^4 \sum_{i,j=1}^{n} m_j\, m_i \left(1 - \frac{1}{2}\, \frac{(u^j - u^i)^2}{c^2} \right) \qquad (3.28)$$

Therefore L_u can be simplified to read

$$L_u = c^2 \sqrt{\sum_{i,j=1}^{n} m_j\, m_i \left(1 - \frac{1}{2}\, \frac{(u^j - u^i)^2}{c^2} \right)} \qquad (3.29)$$

Under a Galilean transformation,

$$x_i' = x_i + V\, t$$

$$L_u' = c^2 \sqrt{\sum_{i,j=1}^{n} m_j\, m_i \left(1 - \frac{1}{2}\, \frac{(u^{j'} - u^{i'})^2}{c^2} \right)} = c^2 \sqrt{\sum_{i,j=1}^{n} m_j\, m_i \left(1 - \frac{1}{2}\, \frac{(u^j - u^i)^2}{c^2} \right)} = L_u \qquad (3.30)$$

since

$$u^{j'} - u^{i'} = (u^j + V) - (u^i + V) = u^j - u^i \qquad (3.31)$$

Particle Density Theorem and fluid paths.

With a discrete number of observables the path of a system
as it traces out the shortest distance in the configuration space can be
described by x (i, t) which is basically the value of observable x_i

at time t and where i is an integer. To elaborate further the path can be thought of as a mapping

$$X[t] : N \rightarrow R_e , \quad \text{where } N = \{1, 2, 3 \; ....\} \qquad (4.1)$$

ie a mapping which changes continually with t. This concept can be extended to the case where i is itself continuous, ie

$$X[t] : S \rightarrow R_e . \text{ where } S = [0 , 1] \qquad (4.2)$$

By allowing the observable label to become continuous we can begin to describe the evolution of a continuum of observations. In the example to follow the system will be a physical fluid confined to move in two dimensions. Throughout the generalization to three dimensions will be apparent.

Let

$$(X(\lambda, \beta, t), \; Y(\lambda, \beta, t)) , \; (\lambda, \beta) \in S * S$$

describe the path of a two dimensional fluid. So the velocity vector of a particle identified by the pair $((\lambda, \beta))$ is simply

$$\left(\frac{\delta X[\lambda, \beta, t]}{\delta t} , \frac{\delta Y[\lambda, \beta, t]}{\delta t} \right) \qquad (4.3)$$

For conceptualization each labeling pair (λ, β), could be, although not necessarily, identified with the initial position of the particles, ie the path could be described as a mapping between where that particles where at time $t = t_0$ and where they are at time t, ie

$$(X_0, \; Y_0, t) \; \rightarrow (X_t, \; Y_t, t) \text{ or } \; (X(X_0, Y_0, t), \; Y(X_0, Y_0, t))$$

However with out any loss of generality for the remainder of the analysis to follow we will assume that

$$(\lambda, \beta) \in S * S$$

By observing that

$$1 = \int d\lambda \, d\beta \; = \int \rho \, dX \, dY \qquad (4.4)$$

where

$$\rho = \text{Det} \begin{pmatrix} \frac{\delta\lambda}{\delta X} & \frac{\delta\lambda}{\delta Y} \\ \frac{\delta\beta}{\delta X} & \frac{\delta\beta}{\delta Y} \end{pmatrix} \qquad (4.5)$$

suggests that ρ, which is a measure of the particle number density, i.e the number of particles per unit volume, at time t, behaves like a conserved quantity and in fact it can be shown that

$$\frac{d\rho}{dt} + \rho\,\nabla.V = 0 \qquad (4.6)$$

where

$$V = \left(\frac{\delta X[\lambda, \beta, t]}{\delta t}, \frac{\delta Y[\lambda, \beta, t]}{\delta t}\right) = (V_x, V_y) \qquad (4.7)$$

$$\nabla = \left(\frac{\delta}{\delta X}, \frac{\delta}{\delta Y}\right)$$

note $\frac{d\rho}{dt}$ is the total derivative of ρ, so it describes the change in ρ as we follow a point being convected along in the fluid, ie for a constant (λ, β)

proof :

$$\rho = \mathrm{Det}\begin{pmatrix} \frac{\delta\lambda}{\delta X} & \frac{\delta\lambda}{\delta Y} \\ \frac{\delta\beta}{\delta X} & \frac{\delta\beta}{\delta Y} \end{pmatrix} = \frac{1}{\mathrm{Det}\begin{pmatrix} \frac{\delta X}{\delta\lambda} & \frac{\delta X}{\delta\beta} \\ \frac{\delta Y}{\delta\lambda} & \frac{\delta Y}{\delta\beta} \end{pmatrix}} = \frac{1}{\rho'} \qquad (4.8)$$

therefore

$$\frac{d\rho}{dt} + \rho\,\nabla.V = \frac{d\,\frac{1}{\rho'}}{dt} + \frac{1}{\rho'}\left(\frac{\delta V_x}{\delta X} + \frac{\delta V_y}{\delta Y}\right)$$

$$= \frac{d\,\frac{1}{\rho'}}{dt} + \frac{1}{\rho'}\left(\frac{\delta V_x}{\delta\lambda}\frac{\delta\lambda}{\delta X} + \frac{\delta V_x}{\delta\beta}\frac{\delta\beta}{\delta X} + \frac{\delta V_y}{\delta\lambda}\frac{\delta\lambda}{\delta X} + \frac{\delta V_y}{\delta\beta}\frac{\delta\beta}{\delta X}\right) \qquad (4.9)$$

Using the fact that

$$\begin{pmatrix} \frac{\delta\lambda}{\delta X} & \frac{\delta\lambda}{\delta Y} \\ \frac{\delta\beta}{\delta X} & \frac{\delta\beta}{\delta Y} \end{pmatrix} = \begin{pmatrix} \frac{\delta X}{\delta\lambda} & \frac{\delta X}{\delta\beta} \\ \frac{\delta Y}{\delta\lambda} & \frac{\delta Y}{\delta\beta} \end{pmatrix}^{-1} = \frac{1}{\rho'}\begin{pmatrix} \frac{\delta Y}{\delta\beta} & -\frac{\delta X}{\delta\beta} \\ -\frac{\delta Y}{\delta\lambda} & \frac{\delta X}{\delta\lambda} \end{pmatrix} \qquad (4.10)$$

and that

$$\frac{d\rho'}{dt} = \frac{\delta}{\delta t}\left(\mathrm{Det}\begin{pmatrix} \frac{\delta X}{\delta\lambda} & \frac{\delta X}{\delta\beta} \\ \frac{\delta Y}{\delta\lambda} & \frac{\delta Y}{\delta\beta} \end{pmatrix}\right) = \frac{\delta}{\delta t}\left(\frac{\delta X}{\delta\lambda}\frac{\delta Y}{\delta\beta} - \frac{\delta X}{\delta\beta}\frac{\delta Y}{\delta\lambda}\right)$$

$$= \frac{\delta \mathrm{X}}{\delta\lambda}\frac{\delta \mathrm{V}_\mathrm{y}}{\delta\beta} + \frac{\delta \mathrm{V}_\mathrm{x}}{\delta\lambda}\frac{\delta \mathrm{Y}}{\delta\beta} - \left(\frac{\delta \mathrm{X}}{\delta\beta}\frac{\delta \mathrm{V}_\mathrm{y}}{\delta\lambda} + \frac{\delta \mathrm{V}_\mathrm{x}}{\delta\beta}\frac{\delta \mathrm{Y}}{\delta\lambda}\right) \qquad (4.11)$$

eq (2) becomes,

$$\frac{\mathrm{d}\,\frac{1}{\rho'}}{\mathrm{dt}} + \frac{1}{\rho'}\left(\frac{\delta \mathrm{V}_x}{\delta\lambda}\frac{\delta\lambda}{\delta \mathrm{X}} + \frac{\delta \mathrm{V}_x}{\delta\beta}\frac{\delta\beta}{\delta \mathrm{X}} + \frac{\delta \mathrm{V}_y}{\delta\lambda}\frac{\delta\lambda}{\delta \mathrm{X}} + \frac{\delta \mathrm{V}_y}{\delta\beta}\frac{\delta\beta}{\delta \mathrm{X}}\right)$$

$$= \quad -\frac{1}{\rho'^2}\frac{\mathrm{d}\rho'}{\mathrm{dt}} + \frac{1}{\rho'^2}\left(\frac{\delta \mathrm{V}_\mathrm{x}}{\delta\lambda}\frac{\delta \mathrm{Y}}{\delta\beta} + \frac{\delta \mathrm{X}}{\delta\lambda}\frac{\delta \mathrm{V}_\mathrm{y}}{\delta\beta} - \frac{\delta \mathrm{V}_\mathrm{x}}{\delta\beta}\frac{\delta \mathrm{Y}}{\delta\lambda} - \frac{\delta \mathrm{X}}{\delta\beta}\frac{\delta \mathrm{V}_\mathrm{y}}{\delta\lambda}\right)$$

$$= \frac{1}{\rho'^2}\left(\frac{\mathrm{d}\rho'}{\mathrm{dt}} - \frac{\mathrm{d}\rho'}{\mathrm{dt}}\right) = 0 \qquad (4.12)$$

i.e

$$\frac{\mathrm{d}\rho}{\mathrm{dt}} + \rho\,\nabla.\mathrm{V} = 0 \qquad (4.13)$$

Null

■ Construction of a modified Electromagnetic Lagrangian from particle pairs

In what will follow it can be demonstrated that the Lorentz force law and the Maxwell non retarded Electromagnetic Potentials can be simultaneously generated from a lagrangian constructed from a single interaction energy between particle pairs .

Consider an ensemble of particles, $i = 1, n$ whose positions and velocities are observed with respect to a Cartesian reference system.

Define the interaction energy between particles i and j as

$$\Psi\,(x_i\,,\, y_i,\, z_i\,,\, x_j\,,\, y_j,\, z_j) \;=\; \Psi_{i,j}$$

and that

$$\Psi_{i,j} \;=\Psi_{j,i} \qquad (5.1)$$

where $(x_i\,,\, y_i,\, z_i)$ and $(x_j\,,\, y_j,\, z_j)$ are the coordinates of particles i and j respectively

Also define the Kinetic Energy, $T_{i,j}$, associated with particles i and j to be

$$T_{i,j} = \frac{1}{2}\,\frac{\Psi_{i,j}}{c^2}\,((u_i + u_j)^2 + (v_i + v_j)^2 + (w_i + w_j)^2) \qquad (5.2)$$

where (u_i, v_i, w_i) and (u_j, v_j, w_j) are the velocities of particles i and j respectively.

The Classical non Relativistic Lagrangian is then taken to be

$$L_c = \Phi - T = \sum_{(i,j)=1}^{n} (\Psi_{i,j} - T_{i,j}) \qquad i \neq j \qquad (5.3)$$

It can be shown (see appendix A) that if a Lagrangian, L, has the symmetric form

$$L = \sum_{i,j=1}^{n} l\,(x_i, y_i, z_i, x_j, y_j, z_j, u_i, v_i, w_i, u_j, v_j, w_j)$$

$$= \sum_{i,j=1}^{n} l_{i,j}\ ,\ l_{i,j} = l_{j,i} \qquad (5.4)$$

then the equations of motion generated by L are;

$$\frac{d}{dt}\left[\frac{\delta L_k}{\delta u_k}\right] - \frac{\delta L_k}{\delta x_k} = 0\,,$$
$$\frac{d}{dt}\left[\frac{\delta L_k}{\delta v_k}\right] - \frac{\delta L_k}{\delta y_k} = 0,$$
$$\frac{d}{dt}\left[\frac{\delta L_k}{\delta w_k}\right] - \frac{\delta L_k}{\delta z_k} = 0 \quad k = 1, n \qquad (5.5)$$

where

$$L_k = \sum_{j=1}^{n} l_{k,j} \qquad j \neq k \text{ and } L = \sum_{k=1}^{n} L_k \qquad \backslash \qquad (5.6)$$

In in the case for L_c

$$L_{c,k} = \Phi_k - \frac{1}{2}\,m_k\,V_k.V_k - A_k.V_k - \frac{1}{2}\,G_k \qquad (5.7)$$

where

$$\Phi_k = \sum_{j=1}^{n} \Psi_{k,j} \qquad (5.8)$$

$$m_k = \frac{1}{c^2}\sum_{j=1}^{n} \Psi_{k,j} \qquad (5.9)$$

$$G_k = \frac{1}{c^2}\sum_{j=1}^{n}\Psi_{k,j}\,(u_j^{\,2} + v_j^{\,2} + w_j^{\,2}) \qquad (5.10)$$

$$V_k = (u_k\ ,\ v_k\ ,\ w_k\ ) \qquad (5.11)$$

$$A_k = (A_{x,k}\ ,\ A_{y,k}\ ,\ A_{z,k}\ ) \qquad (5.12)$$

$$A_{x,k} = \frac{1}{c^2}\sum_{j=1}^{n}\Psi_{k,j}\ u_j \qquad (5.13)$$

$$A_{y,k} = \frac{1}{c^2}\sum_{j=1}^{n}\Psi_{k,j}\ v_j \qquad (5.14)$$

$$A_{z,k} = \frac{1}{c^2}\sum_{j=1}^{n}\Psi_{k,j}\ w_j \qquad (5.15)$$

By inspection it is clear that

$$\Phi_k - \frac{1}{2}\, m_k\, V_k.V_k - A_k.V_k \qquad (5.16)$$

will generate the Lorentz force law. $L_{c,k}$ departs slightly from $\Phi_k - \frac{1}{2} m_k\, V_k.V_k - A_k.V_k$ due to the inclusion of the G_k term which is required if the total Lagrangian, L_c, is to be symmetrical with respect to particle exchange. Also the potentials Φ_k and A_k are automatically generated by L_c and can be interpreted as the discrete version of the non retarded Electromagnetic potentials. The m_k potential is to be interpreted as the mass of the particle, k, which leads to the speculation that mass is electromagnetic in origin. Also the reality of mass would suggest, in light of what has been presented here, that the potentials Φ_k and A_k are in fact real entities as opposed to the popularly held view that their derivatives are the real entities, i.e. the electric and magnetic fields. The new scaler potential, G_k, is generated by the square of the speed of the sources and is considerably smaller than the coulomb potential Φ_k due to the $\frac{1}{c^2}$ factor. The construction of the lagrangian L_c from particle pairs would suggest that the fundamental units of matter are in fact particle pairs rather than individual particles themselves. This would seem to make sense as treating a particle in isolation is meaningless unless there is at least one other in order to provide a point of reference. The above analysis can be repeated for a continuous distribution of observables (see Appendix A − continuous case) or for the case when the set of observables form a continuum or a ' fluid path'. Suppose that we Define the interaction energy between particles λ and β

$$\Psi\,(x_\lambda,\ y_\lambda,\ z_\lambda,\ x_\beta,\ y_\beta,\ z_\beta) = \Psi_{\lambda,\beta} \qquad (5.17)$$

and that

$$\Psi_{\lambda,\beta} = \Psi_{\beta,\lambda} \qquad (5.18)$$

where $(x_\lambda, y_\lambda, z_\lambda)$ and $(x_\beta, y_\beta, z_\beta)$ are the coordinates of particles λ and β respectively.Also define the Kinetic Energy ,

$T_{\lambda,\beta}$, associated with particles λ and β to be

$$T_{\lambda,\beta} = \frac{1}{2}\frac{\Psi_{\lambda,\beta}}{c^2}\left((\dot{x}_\lambda + \dot{x}_\beta)^2 + (\dot{y}_\lambda + \dot{y}_\beta)^2 + (\dot{z}_\lambda + \dot{z}_\beta)^2\right) \qquad (5.19)$$

where $(\dot{x}_\lambda , \dot{y}_\lambda, \dot{z}_\lambda)$ and $(\dot{x}_\beta , \dot{y}_\beta, \dot{z}_\beta)$ are the velocities of particles λ and β respectively .The Newtonian non Relativistic Lagrangian is then taken to be

$$L_c = \Phi - T = \int_{\lambda_1=0}^{1}\int_{\lambda_2=0}^{1}\int_{\lambda_3=0}^{1}\int_{\beta_1=0}^{1}\int_{\beta_2=0}^{1}\int_{\beta_3=0}^{1}(\Psi_{\lambda,\beta} - T_{\lambda,\beta})\, d\lambda_1\, d\lambda_2\, d\lambda_3\, d\beta_1\, d\beta_2\, d\beta_3 \qquad (5.20)$$

In the case for L_c a Lagrangian density , $L_{c,\lambda}$, can be obtained (see Appendix A − continuous case)

$$L_{c,\lambda} = \Phi_\lambda - \frac{1}{2} m_\lambda V_\lambda.V_\lambda - A_\lambda.V_\lambda - \frac{1}{2} G_\lambda \qquad (5.21)$$

where

$$\Phi_\lambda = \int_{\beta_1=0}^{1}\int_{\beta_2=0}^{1}\int_{\beta_3=0}^{1}\Psi_{\lambda,\beta}\, d\beta_1\, d\beta_2\, d\beta_3$$

$$= \int_{\beta_1=0}^{1}\int_{\beta_2=0}^{1}\int_{\beta_3=0}^{1}\Psi_{\lambda,\beta}\,\rho\, dx_\beta\, dy_\beta\, dz_\beta \qquad (5.22)$$

$$m_\lambda = \frac{1}{c^2}\int_{\beta_1=0}^{1}\int_{\beta_2=0}^{1}\int_{\beta_3=0}^{1}\Psi_{\lambda,\beta}\, d\beta_1\, d\beta_2\, d\beta_3$$

$$= \frac{1}{c^2}\int_{\beta_1=0}^{1}\int_{\beta_2=0}^{1}\int_{\beta_3=0}^{1}\Psi_{\lambda,\beta}\,\rho\, dx_\beta\, dy_\beta\, dz_\beta = \frac{\Phi_\lambda}{c^2} \qquad (5.23)$$

$$G_\lambda = \frac{1}{c^2}\int_{\beta_1=0}^{1}\int_{\beta_2=0}^{1}\int_{\beta_3=0}^{1}\Psi_{\lambda,\beta}\left(\dot{x}_\beta^{\,2} + \dot{y}_\beta^{\,2} + \dot{z}_\beta^{\,2}\right) d\beta_1\, d\beta_2\, d\beta_3$$

$$= \frac{1}{c^2}\int_{\beta_1=0}^{1}\int_{\beta_2=0}^{1}\int_{\beta_3=0}^{1}\Psi_{\lambda,\beta}\,\rho\left(\dot{x}_\beta^{\,2} + \dot{y}_\beta^{\,2} + \dot{z}_\beta^{\,2}\right) dx_\beta\, dy_\beta\, dz_\beta \qquad (5.24)$$

$$V_\lambda = (\dot{x}_\lambda , \dot{y}_\lambda, \dot{z}_\lambda) \qquad (5.25)$$

$$A_\lambda = (A_{x,\lambda}, A_{y,\lambda}, A_{z,\lambda}) \qquad (5.26)$$

$$A_\lambda = \frac{1}{c^2}\int_{\beta_1=0}^{1}\int_{\beta_2=0}^{1}\int_{\beta_3=0}^{1}\Psi_{\lambda,\beta}\, V_\beta\, d\beta_1\, d\beta_2\, d\beta_3$$

$$= \frac{1}{c^2}\int_{\beta_1=0}^{1}\int_{\beta_2=0}^{1}\int_{\beta_3=0}^{1}\Psi_{\lambda,\beta}\,\rho\, V_\beta\, dx_\beta\, dy_\beta\, dz_\beta \qquad (5.27)$$

The subscript, λ, can be dropped and the continuous version of the equations expressed explicitly with respect to the (x, y, z, t) space. i . e

$$\rho = \begin{pmatrix} \frac{\delta\beta_1}{\delta X} & \frac{\delta\beta_1}{\delta Y} & \frac{\delta\beta_1}{\delta Z} \\ \frac{\delta\beta_2}{\delta X} & \frac{\delta\beta_2}{\delta Y} & \frac{\delta\beta_2}{\delta Z} \\ \frac{\delta\beta_3}{\delta X} & \frac{\delta\beta_3}{\delta Y} & \frac{\delta\beta_3}{\delta Z} \end{pmatrix}$$

$$\Phi(x, y, z, t) = \int_\Omega \Psi(x, y, z, x_\beta, y_\beta, z_\beta)\, \rho(x_\beta, y_\beta, z_\beta, t)\, \mathrm{d}x_\beta\, \mathrm{d}y_\beta\, \mathrm{d}z_\beta \qquad (5.28)$$

$$m(x, y, z, t) = \frac{1}{c^2} \int_\Omega \Psi(x, y, z, x_\beta, y_\beta, z_\beta)\, \rho(x_\beta, y_\beta, z_\beta, t)\, \mathrm{d}x_\beta\, \mathrm{d}y_\beta\, \mathrm{d}z_\beta \qquad (5.29)$$

$$G(x, y, z, t) = \frac{1}{c^2} \int_\Omega \Psi(x, y, z, x_\beta, y_\beta, z_\beta) \left(\dot{x}_\beta^{\,2} + \dot{y}_\beta^{\,2} + \dot{z}_\beta^{\,2}\right) \rho(x_\beta, y_\beta, z_\beta, t)\, \mathrm{d}x_\beta\, \mathrm{d}y_\beta\, \mathrm{d}z_\beta \qquad (5.30)$$

$$A(x, y, z, t) = \frac{1}{c^2} \int_\Omega \Psi(x, y, z, x_\beta, y_\beta, z_\beta)\, I_\beta\, \mathrm{d}x_\beta\, \mathrm{d}y_\beta\, \mathrm{d}z_\beta \qquad (5.31)$$

$$\frac{\partial \rho(x, y, z, t)}{\partial t} + \nabla.(V(x, y, z, t)\, \rho(x, y, z, t)) = 0 \qquad (5.32)$$

$$A_c = \int_0^T L_c\, \mathrm{d}t \qquad (5.33)$$

(Here A_c represents the action variable and is not to be confused with the vector potential; A (x, y, z, t))

$$L_c = \int_\Omega l(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)\, \rho(x, y, z, t)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \qquad (5.34)$$

$$l(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) = \Phi - \frac{1}{2} m V.V - A.V - \frac{1}{2} G = \Phi' - \frac{1}{2} m V.V - A.V \qquad (5.35)$$

$$\Phi' = \Phi - \frac{1}{2} G \qquad (5.36)$$

$$V = (\dot{x}, \dot{y}, \dot{z}) \qquad (5.37)$$

$$\frac{d}{dt}\left(\frac{\partial l}{\partial \dot{x}}\right) - \frac{\partial l}{\partial x} = 0$$

$$\frac{d}{dt}\left(\frac{\partial l}{\partial \dot{y}}\right) - \frac{\partial l}{\partial y} = 0$$

$$\frac{d}{dt}\left(\frac{\partial l}{\partial \dot{z}}\right) - \frac{\partial l}{\partial z} = 0 \qquad (5.38)$$

where

$$I_\beta = \rho(x_\beta, y_\beta, z_\beta)\, V_\beta \qquad (5.39)$$

is the current density and

$V_\beta = (\dot{x}_\beta, \dot{y}_\beta, \dot{z}_\beta)$. $\rho(x_\beta, y_\beta, z_\beta)$ is to be identified with what has been traditionally thought of as the electric charge density .

Force and momentum

The equations of motion generated by,

$$\frac{d}{dt}\left(\frac{\partial l}{\partial \dot{x}}\right) - \frac{\partial l}{\partial x} = 0$$

$$\frac{d}{dt}\left(\frac{\partial l}{\partial \dot{y}}\right) - \frac{\partial l}{\partial y} = 0$$

$$\frac{d}{dt}\left(\frac{\partial l}{\partial \dot{z}}\right) - \frac{\partial l}{\partial z} = 0$$

in vector notation are,

$$\frac{d(m V)}{dt} = -\nabla\Phi\left(1 - \frac{1}{2}\,\frac{V \,.\, V}{c^2}\right) + \frac{1}{2}\,\nabla G - \frac{\partial A}{\partial t} + V \times (\nabla \times A) = E + V \times B \qquad (5.40)$$

where

$$E = -\nabla\Phi\left(1 - \frac{1}{2}\,\frac{V \,.\, V}{c^2}\right) + \frac{1}{2}\,\nabla G - \frac{\partial A}{\partial t} \approx -\nabla\Phi' - \frac{\partial A}{\partial t} \text{ (for small velocities)} \qquad (5.41)$$

$$B = \nabla \times A \qquad (5.42)$$

In the discrete version of our analysis, if the interaction energy,

$$\Psi(x_i, y_i, z_i, x_j, y_j, z_j)$$

has translational invariance with respect to each of the coordinate axis x, y, z

i . e

$$\Psi(x_i + \Delta X, y_i + \Delta Y, z_i + \Delta Z, x_j + \Delta X, y_j + \Delta Y, z_j + \Delta Z) = \Psi(x_i, y_i, z_i, x_j, y_j, z_j) \qquad (5.43)$$

for arbitrary ΔX, ΔY, ΔZ, then

$$L_c(x_i + \Delta X, y_i + \Delta Y, z_i + \Delta Z, u^i, v^i, w^i, i = 1, n) = L_c(x_i, y_i, z_i, u^i, v^i, w^i, i = 1, n) \qquad (5.44)$$

which implies that,

$$\sum_{i=1}^{n} \frac{\partial L_c}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial L_c}{\partial y_i} = \sum_{i=1}^{n} \frac{\partial L_c}{\partial z_i} = 0 \qquad (5.45)$$

which from the Euler Lagrangian equations gives

$$\begin{aligned} \frac{d}{dt}\sum_{k=1}^{n} \frac{\partial L_c}{\partial u^k} = \frac{d}{dt}\sum_{k=1}^{n} \frac{\partial L_c}{\partial v^k} = \frac{d}{dt}\sum_{k=1}^{n} \frac{\partial L_c}{\partial w^k} \\ = 2\frac{d}{dt}\sum_{k=1}^{n} \frac{\partial L_{c,k}}{\partial u^k} = 2\frac{d}{dt}\sum_{k=1}^{n} \frac{\partial L_{c,k}}{\partial v^k} = 2\frac{d}{dt}\sum_{k=1}^{n} \frac{\partial L_{c,k}}{\partial w^k} = 0 \qquad (5.46) \end{aligned}$$

This may be expressed in a more compact form as

$$\frac{d}{dt}\sum_{k=1}^{n} (m_k V^k + A_k) = 0 \qquad (5.47)$$

which can be further simplified by observing that

$$\begin{aligned} \sum_{k=1}^{n} A_k = \sum_{k=1}^{n}\left(\sum_{j=1}^{n} \frac{\Psi_{k,j}}{c^2} V^j\right) = \sum_{j=1}^{n}\left(\sum_{k=1}^{n} \frac{\Psi_{k,j}}{c^2} V^j\right) \\ = \sum_{j=1}^{n} V^j \left(\sum_{k=1}^{n} \frac{\Psi_{k,j}}{c^2}\right) \\ = \sum_{j=1}^{n} V^j \left(\sum_{k=1}^{n} \frac{\Psi_{j,k}}{c^2}\right) = \sum_{j=1}^{n} m_j V^j = \sum_{k=1}^{n} m_k V^k \qquad (5.48) \end{aligned}$$

to give

$$2\frac{d}{dt}\sum_{k=1}^{n} m_k V^k = 0$$

or

$$\frac{d}{dt}\sum_{k=1}^{n} m_k V^k = 0 \qquad (5.49)$$

which is the conservation of Newtonian momentum

The above analysis can be repeated in an analogous fashion to give the continuous version of eq (5.48);

$$\frac{d}{dt}\int_{\Omega} \rho\, m\, V\, dx\, dy\, dz = 0 \qquad (5.50)$$

The quantity $\rho\, m$ is the mass per unit volume or the mass density $\rho_m = \rho\, m$. Since ρ is a conserved quantity then it is easy to show that

$$\frac{d}{dt}\int_{\Omega} \rho\, m\, V\, dx\, dy\, dz = \int_{\Omega} \rho \frac{d}{dt}(m\, V)\, dx\, dy\, dz = 0. \qquad (5.51)$$

In light of the above result we can define the total force, $F_{\Omega'}$, acting over a subregion Ω' to be ;

$$F_{\Omega'} = \int_{\Omega'} \rho \frac{d}{dt}(m\, V)\, dx\, dy\, dz \qquad (5.52)$$

then

$$F_{\Omega} = 0, \qquad (5.53)$$

i.e. the total force on a close system must be zero. Also using the fact that

$$\frac{d}{dt}(m\, V) = E + V \times B \qquad (5.54)$$

implies that

$$F_{\Omega'} = \int_{\Omega'} \rho\, (E + V \times B)\, dx\, dy\, dz \qquad (5.55)$$

If the region Ω' is small enough so that the integrand $(E + \nabla \times B)$ can be taken outside the integral while resulting in a small loss of accuracy, then

$$F_{\Omega'} = (E + V \times B) \int_{\Omega'} \rho\, dx\, dy\, dz = q\, (E + V \times B) \qquad (5.56)$$

where

$$q = \int_{\Omega'} \rho \, d x \, d y \, d z \qquad (5.57)$$

the total charge contained within Ω'

Electromagnetic energy conservation and the Poynting Theorem

The Hamiltonian corresponding to the discrete version of L_c is

$$H_c = L_c - \sum_{k=1}^{n} \left(u^k \frac{\partial L_c}{\partial u^k} + v^k \frac{\partial L_c}{\partial v^k} + w^k \frac{\partial L_c}{\partial w^k} \right) = L_c - 2 \sum_{k=1}^{n} \left(u^k \frac{\partial L_{c,k}}{\partial u^k} + v^k \frac{\partial L_{c,k}}{\partial v^k} + w^k \frac{\partial L_{c,k}}{\partial w^k} \right) \qquad (5.58)$$

Assuming that $\frac{\partial L_c}{\partial t} = 0$, it is easy to show that

$$\frac{dH_c}{dt} = 0 \qquad (5.59)$$

using the lagrangian equations of motion

Also $\frac{dH_c}{dt}$ can be expanded in terms of $L_{c,k}$ to obtain a measure of the energy per particle in the discrete case and analogously, the energy density in the continuous case

$$\begin{aligned}
\frac{dH_c}{dt} = \frac{dL_c}{dt} - 2 \sum_{k=1}^{n} \Bigg(& u^k \frac{d}{dt} \left(\frac{\partial L_{c,k}}{\partial u^k} \right) + v^k \frac{d}{dt} \left(\frac{\partial L_{c,k}}{\partial v^k} \right) + w^k \frac{d}{dt} \left(\frac{\partial L_{c,k}}{\partial w^k} \right) + \\
& \frac{du^k}{dt} \frac{\partial L_{c,k}}{\partial u^k} + \frac{dv^k}{dt} \frac{\partial L_{c,k}}{\partial v^k} + \frac{dw^k}{dt} \frac{\partial L_{c,k}}{\partial w^k} \Bigg)
\end{aligned}$$

$$\begin{aligned}
= \frac{\partial L_c}{\partial t} + 2 \sum_{k=1}^{n} & \frac{\partial L_{c,k}}{\partial u^k} \frac{du^k}{dt} + \frac{\partial L_{c,k}}{\partial v^k} \frac{dv^k}{dt} + \frac{\partial L_{c,k}}{\partial w^k} \frac{dw^k}{dt} + \frac{\partial L_{c,k}}{\partial x_k} u^k + \frac{\partial L_{c,k}}{\partial y_k} v^k + \frac{\partial L_{c,k}}{\partial z_k} w^k \\
& -2 \sum_{k=1}^{n} \Bigg(u^k \frac{d}{dt} \left(\frac{\partial L_{c,k}}{\partial u^k} \right) + v^k \frac{d}{dt} \left(\frac{\partial L_{c,k}}{\partial v^k} \right) + w^k \frac{d}{dt} \left(\frac{\partial L_{c,k}}{\partial w^k} \right) + \\
& \qquad \frac{du^k}{dt} \frac{\partial L_{c,k}}{\partial u^k} + \frac{dv^k}{dt} \frac{\partial L_{c,k}}{\partial v^k} + \frac{dw^k}{dt} \frac{\partial L_{c,k}}{\partial w^k} \Bigg)
\end{aligned}$$

$$= 2 \sum_{k=1}^{n} \frac{d}{dt} \left(L_{c,k} - \frac{\partial L_{c,k}}{\partial u^k} u^k - \frac{\partial L_{c,k}}{\partial v^k} v^k - \frac{\partial L_{c,k}}{\partial w^k} w^k \right) = 0 \qquad (5.60)$$

or simply

$$\frac{d}{dt}\sum_{k=1}^{n}\left(L_{c,k}-\frac{\partial L_{c,k}}{\partial u^{k}}\,u^{k}-\frac{\partial L_{c,k}}{\partial v^{k}}\,v^{k}-\frac{\partial L_{c,k}}{\partial w^{k}}\,w^{k}\right)=0 \qquad (5.61)$$

If we define

$$E_{k}=L_{c,k}-\frac{\partial L_{c,k}}{\partial u^{k}}\,u^{k}-\frac{\partial L_{c,k}}{\partial v^{k}}\,v^{k}-\frac{\partial L_{c,k}}{\partial w^{k}}\,w^{k} \qquad (5.62)$$

to be the energy per particle, then the total energy E ,

$$E=\sum_{k=1}^{n}E_{k} \qquad (5.63)$$

and

$$\frac{dE}{dt}=0 \qquad (5.64)$$

The analysis can be repeated for the continuous scenario and the corresponding expressions are

$$E_{q}=1-\frac{\partial 1}{\partial \dot{x}}\,\dot{x}-\frac{\partial 1}{\partial \dot{y}}\,\dot{y}-\frac{\partial 1}{\partial \dot{z}}\,\dot{z}=\Phi'+\frac{1}{2}\,m\,V\,.\,V \qquad (5.65)$$

the energy per observable or per unit of charge. The total energy, E, is given by

$$E=\int_{\Omega}\left(\Phi'+\frac{1}{2}\,m\,V\,.\,V\right)\rho\,dx\,dy\,dz \qquad (5.66)$$

where

$$\left(\Phi'+\frac{1}{2}\,m\,V\,.\,V\right)\rho$$

is the energy density

It is instructive to examine the rate of change of energy defined by the charge contained in the subregion Ω'.

Let $E_{\Omega'}$ be the amount of energy defined by the charge contained in Ω'

$$E_{\Omega'}=\int_{\Omega'}\left(\Phi'+\frac{1}{2}\,m\,V\,.\,V\right)\rho\,dx\,dy\,dz \qquad (5.67)$$

Then

$$\frac{dE_{\Omega'}}{dt} = \frac{d}{dt}\int_{\Omega'}\left(\Phi' + \frac{1}{2}\,m\,V\,.\,V\right)\rho\,dx\,dy\,dz$$

$$= \int_{\Omega'}\rho\,\frac{d}{dt}\left(\Phi' + \frac{1}{2}\,m\,V\,.\,V\right)dx\,dy\,dz \qquad (5.68)$$

since ρ is a conserved quantity

Observe that

$$\frac{d}{dt}\left(\Phi' + \frac{1}{2}\,m\,V\,.\,V\right) = \frac{d}{dt}\left(\Phi - \frac{1}{2}\,G + \frac{1}{2}\,m\,V\,.\,V\right) = \frac{d}{dt}\,\Phi\left(1 + \frac{1}{2}\,\frac{V\,.\,V}{c^2}\right) - \frac{1}{2}\,\frac{dG}{dt} =$$

$$\frac{d}{dt}\,\Phi\left(1 + \frac{1}{2}\,\frac{V\,.\,V}{c^2}\right) - \frac{1}{2}\,\frac{dG}{dt} = \frac{\Phi\,V}{c^2}\,.\,\frac{dV}{dt} + \left(1 - \frac{1}{2}\,\frac{V\,.\,V}{c^2} + \frac{V\,.\,V}{c^2}\right)\frac{d\Phi}{dt} - \frac{1}{2}\,\frac{dG}{dt} =$$

$$V.\frac{d}{dt}\left(\frac{\Phi\,V}{c^2}\right) + \left(1 - \frac{1}{2}\,\frac{V\,.\,V}{c^2}\right)\frac{d\Phi}{dt} - \frac{1}{2}\,\frac{dG}{dt} = V.\frac{d}{dt}\left(\frac{\Phi\,V}{c^2}\right) + \left(1 - \frac{1}{2}\,\frac{V\,.\,V}{c^2}\right)\left(\frac{\partial\Phi}{\partial t} + V.\triangledown\Phi\right) - \frac{1}{2}\left(\frac{\partial G}{\partial t} + V.\triangledown G\right)$$

$$= V.\left(\frac{d}{dt}\left(\frac{\Phi\,V}{c^2}\right) + \triangledown\Phi\left(1 - \frac{1}{2}\,\frac{V\,.\,V}{c^2}\right) - \frac{1}{2}\,\triangledown G\right) + \left(1 - \frac{1}{2}\,\frac{V\,.\,V}{c^2}\right)\frac{\partial\Phi}{\partial t} - \frac{1}{2}\,\frac{\partial G}{\partial t}$$

$$= V.\left(-\frac{\partial A}{\partial t} + V\times(\triangledown\times A)\right) + \left(1 - \frac{1}{2}\,\frac{V\,.\,V}{c^2}\right)\frac{\partial\Phi}{\partial t} - \frac{1}{2}\,\frac{\partial G}{\partial t} = -V.\frac{\partial A}{\partial t} + \left(1 - \frac{1}{2}\,\frac{V\,.\,V}{c^2}\right)\frac{\partial\Phi}{\partial t} - \frac{1}{2}\,\frac{\partial G}{\partial t}$$

$$\approx -V.\frac{\partial A}{\partial t} + \frac{\partial\Phi}{\partial t} - \frac{1}{2}\,\frac{\partial G}{\partial t} \approx -V.\frac{\partial A}{\partial t} + \frac{\partial\Phi}{\partial t} \qquad \left(\text{if } \frac{V\,.\,V}{c^2} << 1 \text{ and } G << \Phi\right) \qquad (5.69)$$

Also since $\frac{1}{c^2}\,\frac{\partial\Phi}{\partial t} + \triangledown.A = 0$ then eq (5.68) becomes

$$\frac{d}{dt}\left(\Phi' + \frac{1}{2}\,m\,V\,.\,V\right) \approx -V.\frac{\partial A}{\partial t} - c^2\,\triangledown.A\ . \qquad (5.70)$$

Therefore

$$\frac{dE_{\Omega'}}{dt} \approx -\int_{\Omega'}\left(V.\frac{\partial A}{\partial t} + c^2\,\triangledown.A\right)\rho\,dx\,dy\,dz = -\int_{\Omega'}\left(I.\frac{\partial A}{\partial t} + c^2\,\rho\,\triangledown.A\right)dx\,dy\,dz \qquad (5.71)$$

where $I = \rho\,V$ is the current density.

In conclusion judging from the form of eq (5.71) that it is the vector potential which provides the primary mechanism by which energy is exchanged between different regions of an electromagnetic system. The poynting theorem is an in complete statement of energy conservation as it only relates changes in the electromagnetic field to changes in a system' s kinetic energy, i.e. let $T_{\Omega'}$ and $U_{\Omega'}$ denote the classical Newtonian kinetic and potential energies respectively carried by the charge confined to the sub region Ω' so that

$$T_{\Omega'} = \int_{\Omega'} \frac{1}{2}\, m\, V . V\, \rho\, dx\, dy\, dz \qquad (5.72)$$

$$U_{\Omega'} = \int_{\Omega'} \Phi'\, \rho\, dx\, dy\, dz \qquad (5.73)$$

The poynting theorem is easily derived from Maxwell' s equations which are equivalent to the retarded version of eq (5.31) and eq (5.38) it states that

$$\frac{dT_{\Omega'}}{dt} = -\int_{\Omega'} \left(\frac{\partial}{\partial t} (E.E + B.B) + \nabla.(E \times B) \right) dx\, dy\, dz \qquad (5.74)$$

Therefore from the above analysis

$$\frac{dU_{\Omega'}}{dt} - \int_{\Omega'} \left(\frac{\partial}{\partial t} (E.E + B.B) + \nabla.(E \times B) \right) dx\, dy\, dz$$
$$= -\int_{\Omega'} I. \frac{\partial A}{\partial t} + c^2 \rho \nabla.A \;\; dx\, dy\, dz \qquad (5.75)$$

It is only when we consider the entire system do we get equality between the two expressions for the potential energies , or more precisely over the entire region Ω

$$\frac{dU_{\Omega}}{dt} - \int_{\Omega} \left(\frac{\partial}{\partial t} (E.E + B.B) + \nabla.(E \times B) \right) dx\, dy\, dz$$
$$= -\int_{\Omega} \left(I. \frac{\partial A}{\partial t} + c^2 \rho \nabla.A \right) dx\, dy\, dz = \frac{dE_{\Omega}}{dt} = \frac{dE}{dt} = 0 \qquad (5.76)$$

Retarded Potentials

The non retarded nature of the potentials inherent in eqs (5.28) – (5.31) are due to the way measurements are observed in the configuration space of observations. In particular in this formulation of mechanics all particle positions are tabulated against the readings from a single clock , t , which is part of the experimental apparatus employed by the experimenter and which is in the immediate vicinity of the observer / experimenter. If instead of using a single clock which is local to the observer , we use an array of clocks placed at all points in our cartesian reference convention, and all initially synchronized with each othe and to the observer' s local clock, to collectively provide a global time, then measurements made against this global time array will give rise to the retarded form of eq (5.38) and eq (5.31) and hence to Maxwell' s wave equations .

This transition can be made by observing how the charge density obtained from a fluid path defined against the observer' s local time , t, relates to the charge density obtained from a fluid path defined against the global time, t'.

let,

$(x[\lambda_1, \lambda_2, \lambda_3, t], y[\lambda_1, \lambda_2, \lambda_3, t], z[\lambda_1, \lambda_2, \lambda_3, t])$

be the fluid path as measured by the observer using the local time t, and let

$(x'[\lambda_1, \lambda_2, \lambda_3, t'], y'[\lambda_1, \lambda_2, \lambda_3, t'], z'[\lambda_1, \lambda_2, \lambda_3, t'])$

be the fluid path as defined using the global time t'. For a given $(\lambda_1, \lambda_2, \lambda_3)$ the two points are identical when

$$t' = t - \frac{|x - x_o, x - x_o, x - x_o|}{c} \qquad (5.77)$$

or more formally as a coordinate transformation,

$$\begin{aligned} x' &= x \\ y' &= y \\ z' &= z \\ t' &= t - \frac{|x - x_o, y - y_o, z - z_o|}{c} \end{aligned} \qquad (5.78)$$

where c is the speed of light and

(x_o, y_o, z_o)

is the location of the observer. In other words when a fluid particle is at (x, y, z) at time t' as measured on a clock positioned at (x, y, z), it will not be until time t as measured on the observer's clock that its position will be observed.

This implies that the respective charge densities are related by,

$$\rho'[x, y, z, t'] = \rho[x, y, z, t] \qquad (5.79)$$

or

$$\rho[x, y, z, t] = \rho'\left[x, y, z, t - \frac{|x - x_o, y - y_o, z - z_o|}{c}\right] \qquad (5.80)$$

Substituting ρ' in eq (5.28) – eq (5.31) gives rise to the wave equations for the electromagnetic four potential

Interaction energy and curvature of the configuration space

In the analysis presented above, all of the known Electromagnetic phenomenon can be obtained from a lagrangian based on a single interaction energy between particles pairs. However at no point in the above presentation was it alluded to as to what the form of this interaction energy might assume. It can be demonstrated, at least for a configuration space consisting of a single pair, that the interaction energy arises from the configuration space having stationary Riemann curvature. Consider a configuration space consisting of two particles with one particle centered on the origin of a cartesian reference system. The configuration space can be represented by the Riemann metric,

$$M = \Psi[x,\ y,\ z,\ t]^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{c^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{c^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{c^2} \end{pmatrix} \qquad (5.81)$$

as this corresponds to the complete distance element lagrangian

$$L = \sqrt{\Psi^2 - 2\,\Psi\,T} \; = \sqrt{\Psi^2 - \frac{\Psi^2}{c^2}\,(u^2 + v^2 + w^2)} \qquad (5.82)$$

where $(x,\ y,\ z)$ and $(u,\ v,\ w)$ are respectively the position and velocity of the free particle

let

$R_m[x, y, z, t]$

be the Riemann curvature at $(x,\ y,\ z,\ t)$ in the metric space M, and let

$$G = \mathrm{Det}[M] \qquad (5.83)$$

then solving the variational problem

$$\delta \int \sqrt{-G}\; R_m \; dx\, dy\, dz\, dt \; = 0 \qquad (5.84)$$

yields

$$\frac{\partial^2 \Psi}{\partial^2 x} + \frac{\partial^2 \Psi}{\partial^2 y} + \frac{\partial^2 \Psi}{\partial^2 z} - \frac{1}{c^2}\,\frac{\partial^2 \Psi}{\partial^2 t} = 0 \qquad (5.85)$$

If it is assumed that Ψ only depends only on the separation between the particles then

$$\frac{\partial^2 \Psi}{\partial^2 t} = 0$$

and

$$\frac{\partial^2 \Psi}{\partial^2 x} + \frac{\partial^2 \Psi}{\partial^2 y} + \frac{\partial^2 \Psi}{\partial^2 z} = 0 \qquad (5.86)$$

which has the solution

$$\Psi = \frac{Q}{\sqrt{x^2 + y^2 + z^2}} \qquad (5.87)$$

where Q is a constant

in other words Ψ is just the coulomb interaction. More work needs to be done in order to demonstrate that this is true in the more general case of a space consisting of an arbitrary number of particles and also in the continuous case of a fluid path, but it is here conjectured that the interaction energy between an arbitrary pair of observables, i and j is

$$\Psi_{i,j} = \frac{Q}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}} \qquad (5.88)$$

In summary from this perspective, an electromagnetic system can be viewed as a single point moving on a multi dimensional Riemann Manifold of least Riemann curvature

Electromagnetism as a spacetime distortion

The form of eq (5.35) suggests that electromagnetism may have a direct space time representation, where the underlying 4 D Riemann manifold is represented by a metric constructed from the four potential. The structure of the lagrangian density, eq (5.35), and which generates the Lorentz force law, has, at first glance, the appearance of a distance element in a 4 D space time as described by the following metric

$$\begin{pmatrix} \phi & -\frac{A1}{2} & -\frac{A2}{2} & -\frac{A3}{2} \\ -\frac{A1}{2} & -\frac{\phi}{2 c^2} & 0 & 0 \\ -\frac{A2}{2} & 0 & -\frac{\phi}{2 c^2} & 0 \\ -\frac{A2}{2} & 0 & 0 & -\frac{\phi}{2 c^2} \end{pmatrix} \qquad (6.1)$$

However eq (5.35) is strictly speaking, not a distance element lagrangian as it does not contain a square root term. In order to convert eq (5.35) to a proper distance element it has to be assumed that eq (5.35) approximates the lagrangian given by,

$$L = \sqrt{\phi^2 - 2\,\phi T} \approx \phi - T = \phi - \frac{1}{2}\,\frac{\phi}{c^2}\,V.V - A.V \qquad (6.2)$$

$$\left(\text{where } T = \frac{1}{2}\,\frac{\phi}{c^2}\,V.V + A.V\right),$$

and which is represented by the metric;

$$M = \phi \begin{pmatrix} \phi & -A1 & -A2 & -A3 \\ -A1 & -\frac{\phi}{c^2} & 0 & 0 \\ -A2 & 0 & -\frac{\phi}{c^2} & 0 \\ -A2 & 0 & 0 & -\frac{\phi}{c^2} \end{pmatrix} \qquad (6.3)$$

Therefore in the context of a spacetime representation, the Lorentz force law is to be regarded as nothing more than an approximation to a geodesic flow in the spacetime given by eq (6.3). It in instructive to note that eq (6.2) is devoid of any reference to the charge density , ρ, which would seem to suggest that even in the absence of charge, the velocity field is still well defined and has a computable value and would indicate that what we traditionally think of as empty space or the vacuum is in fact imbued with some kind of flow field which merges in a seamless fashion and continuously with the flow of actual charge. The function of charge in this formulation of mechanics serves only to define the field potentials and to give meaning to what we understand as force, momentum and energy.

Scaler charge density and field equations

eq (5.34) effectively demonstrates that eq (5.40) can be obtained from the action

$$A = \int_\Omega l\, d\lambda_1\, d\lambda_2\, d\lambda_3\, dt = \int_\Omega l\rho\, dx\, dy\, dz\, dt \qquad (6.4)$$

where l is the lagrangian density given by

$$l = \phi - \frac{1}{2}\frac{\phi}{c^2} V.V - A.V. \qquad (6.5)$$

It is desirable to express equation (6.4) in a form which is Independent of the particular frame of reference in which it is defined . For this to be possible it is necessary that l and ρ have the characteristics of being a scaler and a scaler density respectively . This can be achieved by conjecturing hat l in this case approximates the distance element lagrangian;

$$l = \sqrt{\phi^2 - \frac{\phi^2}{c^2} V.V - 2\phi A.V.} \qquad (6.6)$$

and observing that,

$$\begin{aligned} A = \int_\Omega l\, d\lambda_1\, d\lambda_2\, d\lambda_3\, dt &= \int_\Omega \left(l \frac{dt}{d\tau}\right) d\lambda_1\, d\lambda_2\, d\lambda_3\, d\tau \\ &= \int_\Omega l'\, d\lambda_1\, d\lambda_2\, d\lambda_3\, d\tau = \int_\Omega l'\rho'\, dx\, dy\, dz\, dt \qquad (6.7) \end{aligned}$$

where $d\tau$ is the scaler quantity (more commonly referred to as the proper time in Relativity theory) defined by

$$d\tau = dt\sqrt{\phi^2 - \frac{\phi^2}{c^2}\,V.V - 2\,\phi\,A.V} \quad , \qquad (6.8)$$

and

$$1' = 1\,\frac{dt}{d\tau} = 1 \qquad (6.9)$$

and which is therefore a scaler, and

$$\rho' = \frac{\rho}{\frac{dt}{d\tau}}$$, a true scaler density

As it stands that action quantity , eq (6.4), is only capable of generating the geodesic equations, which in this case approximate the lorentz force law. In order to concurrently generate the field equations, ie Maxwell' s equations, it is necessary to include an extra term which involves higher derivatives in the components of the metric tensor. It is felt that the most natural route to achieving this is to use the same approach as that described in the Einstein – Hilbert action, which is to extend A to include the 4 D curvature , i.e

let

$$A = \int_\Omega \left(\sqrt{-G}\ R + \lambda\, 1'\rho'\right) dx\,dy\,dz\,dt \qquad (6.10)$$

where G = Det[M] , R is the Riemann curvature of the space described by M and λ is a constant to be determined.

If we let $g_{i,j}$ represent the components of M, then employing the Einstein summation convention

$$1' = \sqrt{g_{i,j}\ v^i v^j} = \sqrt{g^{i,j}\ v_i v_j} \qquad (6.11)$$

where

$$(v^i) = \left(\frac{dt}{d\tau}, \frac{dt}{d\tau}\,V\right), \qquad (6.12)$$

is the four velocity .

Subjecting A to the variational process yields

$$\delta A = \int_\Omega \left(\sqrt{-G}\ E^{i,j} + \lambda\,\rho'\,v^i v^j\right)\delta g_{i,j}\ \ dx\,dy\,dz\,dt \qquad (6.13)$$

where $E^{i,j}$ is the Einstein tensor.

In the particular case of M above, the $g_{i,j}$ components are not independent and therefore cannot be flexed independently which is why we cannot individually set each of the terms in the integrand to zero. To elaborate further;

$$g_{0,0} = \phi^2, \; g_{1,1} = g_{2,2} = g_{3,3} = -\frac{g_{0,0}}{c^2}$$

$$g_{0,i} = g_{i,0} = -\phi A_i, \quad i = 1, 2, 3 \qquad (6.14)$$

For the purpose of illuminating the feasibility of this approach the problem will be simplified by neglecting the magnetic potential, i.e. it will be assumed that $g_{0,i} = 0$, i = 1, 2, 3

then the variational problem becomes

$$0 = \delta A = \int_\Omega \left(\sqrt{-G}\; E^{0,0} + \lambda \rho' (v^0)^2\right)\delta g_{0,0} - \left(\sqrt{-G}\; E^{1,1} + \lambda \rho' (v^1)^2\right)\frac{\delta g_{0,0}}{c^2}$$
$$-\left(\sqrt{-G}\; E^{2,2} + \lambda \rho' (v^2)^2\right)\frac{\delta g_{0,0}}{c^2} - \left(\sqrt{-G}\; E^{3,3} + \lambda \rho' (v^3)^2\right)\frac{\delta g_{0,0}}{c^2}\, dx\, dy\, dz\, dt \qquad (6.15)$$

The variation, $\delta g_{0,0}$, is arbitrary which implies that

$$\sqrt{-G}\left(E^{0,0} - \frac{E^{1,1}}{c^2} - \frac{E^{2,2}}{c^2} - \frac{E^{3,3}}{c^2}\right) + \lambda \rho' \left((v^0)^2 - \frac{(v^1)^2}{c^2} - \frac{(v^2)^2}{c^2} - \frac{(v^3)^2}{c^2}\right) = 0 \qquad (6.16)$$

which is equivalent to

$$\sqrt{-G}\left(E^{0,0} - \frac{E^{1,1}}{c^2} - \frac{E^{2,2}}{c^2} - \frac{E^{3,3}}{c^2}\right) + \lambda v^0 \rho \left(1 - \frac{1}{c^2}(u^2 + v^2 + w^2)\right) =$$

$$\sqrt{-G}\left(E^{0,0} - \frac{E^{1,1}}{c^2} - \frac{E^{2,2}}{c^2} - \frac{E^{3,3}}{c^2}\right) + \lambda \frac{\rho}{\phi}\sqrt{1 - \frac{1}{c^2}(u^2 + v^2 + w^2)} = 0. \qquad (6.17)$$

Now based on

$$M = \phi^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{c^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{c^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{c^2} \end{pmatrix},$$

it is straight forward, although somewhat tedious, to show that

$$\sqrt{-G}\left(E^{0,0}-\frac{E^{1,1}}{c^2}-\frac{E^{2,2}}{c^2}-\frac{E^{3,3}}{c^2}\right)=-\frac{1}{\phi}\left(6\,\frac{1}{c^3}\left(c^2\left(\frac{\partial^2\Phi}{\partial^2 x}+\frac{\partial^2\Phi}{\partial^2 y}+\frac{\partial^2\Phi}{\partial^2 z}\right)-\frac{\partial^2\Phi}{\partial^2 t}\right)\right) \qquad (6.18)$$

which using eq (6.17), gives

$$-\frac{1}{\phi}\left(6\,\frac{1}{c^3}\left(c^2\left(\frac{\partial^2\Phi}{\partial^2 x}+\frac{\partial^2\Phi}{\partial^2 y}+\frac{\partial^2\Phi}{\partial^2 z}\right)-\frac{\partial^2\Phi}{\partial^2 t}\right)\right)+\lambda\,\frac{\rho}{\phi}\sqrt{1-\frac{1}{c^2}\,(u^2+v^2+w^2)}=0 \qquad (6.19)$$

or

$$\frac{\partial^2\Phi}{\partial^2 x}+\frac{\partial^2\Phi}{\partial^2 y}+\frac{\partial^2\Phi}{\partial^2 z}-\frac{1}{c^2}\,\frac{\partial^2\Phi}{\partial^2 t}$$

$$=\frac{\lambda\,\rho\,c}{6}\sqrt{1-\frac{1}{c^2}\,(u^2+v^2+w^2)}\approx\frac{\lambda\,\rho\,c}{6}\left(1-\frac{1}{2\,c^2}\,(u^2+v^2+w^2)\right) \qquad (6.20)$$

if λ is chosen such that $\frac{\lambda\,\rho\,c}{6}=\rho$, i.e. $\lambda=\frac{6}{c}$, then

$$\frac{\partial^2\Phi}{\partial^2 x}+\frac{\partial^2\Phi}{\partial^2 y}+\frac{\partial^2\Phi}{\partial^2 z}-\frac{1}{c^2}\,\frac{\partial^2\Phi}{\partial^2 t}\approx\rho\left(1-\frac{1}{2\,c^2}\,(u^2+v^2+w^2)\right). \qquad (6.21)$$

However if from the outset we do not impose any particular structure or form on what M should be and allow each of the metric components, $g_{i,j}$, to be flexed independently and freely at will, then $\delta A=0$ produces the following field equations

$$\sqrt{-G}\;E^{i,j}+\lambda\,\rho'\,v^i\,v^j=0 \qquad (6.22)$$

or

$$E^{i,j}=-\frac{\lambda\,\rho'}{\sqrt{-G}}\,v^i\,v^j \qquad (6.23)$$

(also take note that the quantity, $\frac{\lambda\,\rho'}{\sqrt{-G}}$, is a scaler)

which number ten equations in ten unknown metric components. Eq (6.23) are basically the Einstein field equations with the energy – momentum tensor,

$$T^{i,j} = \frac{\rho'}{\sqrt{-G}} v^i v^j . \qquad (6.24)$$

The difference, which is something that will be explored in the following section, however lies in the interpretation of the components of the metric tensor. In this theory the metric components, to an approximation, are being identified directly with the entities which define the electromagnetic field which as we have already seen from eq (5.35), is a viable and workable concept. Emphasis should be placed on the term 'approximation', for it is to be asserted that Maxwell's equations represent only an approximation to equation eq (6.23) and that eq (6.23) represent a larger domain of phenomenon other than what can be described by Maxwell's equations. In fact it is to be inferred that eq (6.23) is a description of all physical reality, in that physical reality, that is electromagnetism, gravity matter and whatever other forces that may manifest themselves at the atomic level, is nothing more than a distortion of the spacetime continuum .From the field equations, observing that the Einstein tensor is identically divergence free, the equations of motion are also generated, and incidentally without having to assume geodesic motion, i.e,

$$\begin{aligned}&\left(\sqrt{-G}\ E^{i,j} + \lambda \rho' v^i v^j\right); j \\ &\quad = \sqrt{-G}\ E^{i,j}; j + E^{i,j} \sqrt{-G}\ ; j + (\rho' v^i v^j); j \\ &\qquad = 0 + 0 + (\rho' v^i v^j); j = 0 \qquad (6.25)\end{aligned}$$

since $\sqrt{-G}\ ; j = 0$

Perception of spacetime distortion

In the analysis presented in the last section, the viability of physical phenomenon as a disturbance in spacetime was demonstrated Yet at no point was it explored as to how an experimenter might directly measure such disturbances and how they might manifest themselves as perceived by an observer. In a variety of expositions on Riemann Geometry, the scenario is always given as to how, for example a surveyor, armed with only a measuring rod which defines a unit length might determine the metric tensor describing the characteristics of a two dimensional surface. Firstly a surveyor does not need a measuring rod to ascertain the qualitative characteristics of the 2 D surface as he can directly perceive them having the advantage of existing in a three dimensional space and being able to observe the embedded 2 D surface.

However even if the surveyor was condemned to exist in two dimensions, armed with only a single measuring rod the quantitative properties of his 2 D flatland could be determined. Firstly a coordinate system, (x_1, x_2) is established and then, systematically the end points of the measuring rod are always claimed to be the same distance apart, which for convenience can be chosen to be one, regardless of where on the surface the rod is placed and regardless of its orientation. So

$$1 = g_{1,1} (x_1, x_2) \Delta x_1^2 + 2 g_{1,2} (x_1, x_2) \Delta x_1 \Delta x_2 + g_{2,2} (x_1, x_2) \Delta x_2^2 g_{2,2} \qquad (6.26)$$

where $(\Delta x_{1,} \Delta x_2)$ are the differences in the coordinates of the end points of the measuring rod which has one of its end points placed at (x_1, x_2). The surveyor would then rotate the rod about (x_1, x_2) to obtain several pairs of $(\Delta x_{1,} \Delta x_2)$,

$(\Delta x_1{}^i , \Delta x_2{}^i)$, i = 1, n

to provide a set of equations

$$1 = g_{1,1}(x_1, x_2)(\Delta x_1{}^i)^2 + 2\, g_{1,2}(x_1, x_2)(\Delta x_1{}^i)(\Delta x_2{}^i) + g_{2,2}(x_1, x_2)(\Delta x_2{}^i)^2\, g_{2,2} \quad , \ i = 1, n, \qquad (6.27)$$

which could be used to produce a good estimate for

$$\begin{pmatrix} g_{1,1}(x_1, x_2) & g_{1,2}(x_1, x_2) \\ g_{1,2}(x_1, x_2) & g_{2,2}(x_1, x_2) \end{pmatrix}. \qquad (6.28)$$

The process is then repeated may times at different points until the entire surface has been covered . This process will provide a numerical description of the properties of the surface, from which the curvature can be calculated and which is independent of the choice of coordinate system. However the question still remains as to how a curved 2 D surface might be perceived by a flatlander. The answer to this question lies in the choice of the coordinate system. In the mind of a flatlander a cartesian convention is always used, that is the mental model used to project and imagine space is a flat Euclidean plane, regardless of how curved the space it really is, the points in which can be addressed with the use of two virtual perpendicular axis' s .. Experimentally such a coordinate system could be constructed using the following procedure. The surveyor remains in one position, point A, and employs an assistant at a neighboring point, point B, and which is at a known least distance from point A according to the measuring rod which defines a unit of distance. In order to address the points in the space, each surveyor measures the angle between the particular point being observed and the observed position of his colleague, providing two angles , (α_1, α_2) . Then employing a Euclidean convention (triangulation), the mental model of the observer, and the distance between A and B, the cartesian coordinates (x, y) of the observed point are calculated .. It has to be emphasized that using a Euclidean convention to construct such a coordinate system no way provides any information as to whether the space really is Euclidean or not, it simply provides a convenient and intuitive device to map the points on the surface which corresponds to the mental model of the observer. The procedure to calculate the metric tensor can be implemented by employing a third assistant to move around the space placing the measuring rod at different positions and in different orientations, while at each instant the coordinates of the endpoints are observed by surveyors A and B .The displacements of the endpoints $(\Delta x, \Delta y)$, are then used to calculate the metric tensor by asserting that the measuring rod is always of unit length. i.e

$$1 = g_{x,x}(x, y)\,\Delta x^2 + 2\, g_{x,y}(x, y)\,\Delta x\,\Delta y + g_{y,y}(x, y)\,\Delta y^2 \qquad (6.29)$$

Generally speaking the apparent length , $d = \Delta x^2 + \Delta y^2$, will not be equal to one , i.e,

$$d \neq 1$$

In other words the main qualifying feature of a departure of the surface from a flaplane will be perceived by a flatlander as the apparent stretching or contraction of the measuring rod, the standard unit of length. In the same fashion a departure from a flat Euclidean space in a three dimensional world , our world, would manifest itself as the contraction or stretching of objects. Experimentally distance is a well defined concept in that the distance between two points is a measure of the least number of identical measuring rods needed to span the two points. Utilizing this notion provides a concrete means of identifying the distortion of objects as stress' s and strains. Imagine a configuration of three identical measuring rods arranged such that they form a triangle, and then picture the triangle being moved to a region of space where the metric tensor dictated that the distances between the vertices of the triangle were no longer equal , i.e. the number of measuring rods needed to span each pair of vertices was no longer one in all cases. The implication here would be that the triangle could no longer exist in its original configuration in this new region of space, which meant that at some point during its journey it would have to break hence establishing the presence of a force which is just a manifestation of the space' s non Euclidean nature. At this point emphasis should be placed on the fact that the measuring rods are not abstract entities but real physical objects. It should be emphasized that in order to be able to define 3 D space as a real physical entity, the behavior of a measuring rod should be independent of the material from which it is constructed in the limit that the material itself does not significantly contribute or alter the quantitative properties of the space . So for example a measuring rod constructed from steel would exhibit the same behavior in terms of stretching or contracting as a rod made from copper.

Otherwise t would be difficult to assign any physical meaning or definition to the real nature of space since it could be argued that the behavior of the measuring rods was more attributable to their constituents rather than having anything to do with a non Euclidean space. In order to extend these ideas to a four dimensional world of time and space we must first rigorously define what a spacetime is and how it can be quantified from the viewpoint of practical experiment. A space time event (x, y, z, t) will simply be defined as the point (x, y, z) being observed at time t, where t is registered by a single clock in the locality of the experimenter. The 3 D points are mapped using the same procedure described above, i.e. unitizing surveyors and a Euclidean convention . The 4 D distance element is

$$\Delta s^2 = g_{0,0}\,\Delta t^2 + 2\sum_{i=1}^{3} g_{0,i}\,\Delta t\,\Delta x_i + \sum_{i,j=1}^{3} g_{i,j}\,\Delta x_i\,\Delta x_j \qquad (6.30)$$

where

$(x_1, x_2, x_3) = (x, y, z)$.

For $\Delta t = 0$, Δs is pure space like which allows the ($g_{i,j}$, i = 1, 3, j = 1, 3) components to be calculated in the manner described above , that is using a measuring rod to calibrate the metric components over the entire 3 D region.

Since the components need to be calibrated instantly at time t then to be practical many observations using many measuring rods would need to be taken simultaneously. For $\Delta x_i = 0$, Δs is pure time like and

$$\Delta s^2 = g_{0,0}\,\Delta t^2 \qquad (6.30 .1)$$

Given equation (6.30 .1), the question becomes how can $g_{0,0}$ (x, y, z, t) be measured . A measuring rod cannot be used as spatial length does not have meaning in this context. Instead of a measuring rod what must be used is the tick of a clock to define a standard unit of time. For convenience it can be assumed and without loss of generality that the observer is located at the origin of the coordinate system, the construction of which has been described above.

Let

$$g_{0,0}(0, 0, 0, t) = g_{0,0}{}' \qquad \backslash \quad (6.31)$$

be the value of $g_{0,0}$ at the origin at time t as registered by a clock in the locality of the experimenter , or for clarity, a clock which is located at the origin . Let t' be the time registered by a clock which is identical to, and has been synchronized with the clock located at the origin and which has been then subsequently relocated to (x, y, z).

The hypothesis to be adopted here is that

$$\Delta s^2 = g_{0,0}{}'\,\Delta t'^2 \qquad (6.32)$$

In other words the clock identical to the experimenter' s origin clock and which is then sent to (x, y, z) is to be employed in the same analogous capacity for determining the time like quantity, $g_{0,0}$ (x, y, z, t), as the measuring rod was for determining the pure space like quantities , ($g_{i,j}$, i = 1, 3, j = 1, 3). Therefore

$$g_{0,0}{}' \Delta t'^2 = g_{0,0} \Delta t^2 \qquad (6.33)$$

or

$$\frac{\Delta t'}{\Delta t} = \sqrt{\frac{g_{0,0}}{g_{0,0}{}'}} \qquad (6.34)$$

indicating that the clocks will run at different rates depending on the ratio $\frac{g_{0,0}}{g_{0,0}{}'}$. $g_{0,0}{}' \Delta t'^2$ defines a standard time like length which can be used to calculate the relative value of $g_{0,0}$ (x, y, z, t) at all points in the 4 D space.

It is the ratio of the observed clock rates which allow for the measurement of $\frac{g_{0,0}}{g_{0,0}{}'}$.If during the course of experiments the observer finds the result to be independent of the physical character and design of the clocks, then it is to be concluded in all probability that time is a real physical entity , since it could otherwise be argued that the results were due to spurious interference effects caused by the differences in the potentials, $g_{0,0}$ and $g_{0,0}{}'$, on the rhythm of the clocks and which varied depending on the clocks particular construction rather than been the result of a real, yet intangible temporal property of the universe. Unlike Relativity theory where time dilation and length distortion are regarded as predictions of the theory, in this exposition they are to be viewed as the primary physical phenomenon which are directly unitized by an experimenter to uniquely infer the components of the metric tensor. In fact there are no other means by which the metric tensor can be measured, and should they not exist then spacetime would be strictly Euclidean and devoid of all the rich and varied structures which make the universe the way it is.

Velocity induced time dilation and fluid paths

Consider a fluid Path

$$(x[\lambda_1, \lambda_2, \lambda_3, t], y[\lambda_1, \lambda_2, \lambda_3, t], z[\lambda_1, \lambda_2, \lambda_3, t])$$

If for the moment it is assumed that the metric tensor , $g_{i,j}$, in the (x, y, z, t) space is known then since a fluid path is effectively a transformation from one coordinate system , the $(\lambda_1, \lambda_2, \lambda_3, t)$ space, to another , the (x, y, z, t) space, then it is possible to calculate the metric tensor in the particle labeling space, $(\lambda_1, \lambda_2, \lambda_3, t)$, and which will be denoted by $G_{i,j}$. The distance elements in both spaces are identical so

$$\begin{aligned} \Delta s^2 &= g_{0,0} \Delta t^2 + 2 \sum_{i=1}^{3} g_{0,i} \Delta t \Delta x_i + \sum_{i,j=1}^{3} g_{i,j} \Delta x_i \Delta x_j \\ &= G_{0,0} \Delta t^2 + 2 \sum_{i=1}^{3} G_{0,i} \Delta t \Delta\lambda_i + \sum_{i,j=1}^{3} G_{i,j} \Delta\lambda_i \Delta\lambda_j \qquad (6.35) \end{aligned}$$

where

$(x_1, x_2, x_3) = (x, y, z)$.

$G_{0,0}[\lambda_1, \lambda_2, \lambda_3, t]$ can be calculated by utilizing the particular displacement where $\Delta\lambda_i = 0$. i.e

$$g_{0,0}\, \Delta t^2 + 2 \sum_{i=1}^{3} g_{0,i}\, \Delta t\, \Delta x_i + \sum_{i,j=1}^{3} g_{i,j}\, \Delta x_i\, \Delta x_j = G_{0,0}\, \Delta t^2 \qquad (6.36)$$

in which case

$$\frac{\Delta x_i}{\Delta t} = u^i \,,$$

the velocity of the fluid particle labeled by $(\lambda_1, \lambda_2, \lambda_3)$. So

$$g_{0,0}\,[x, y, z, t] + 2 \sum_{i=1}^{3} g_{0,i}\,[x, y, z, t]\, u^i + \sum_{i,j=1}^{3} [x, y, z, t]\, u^i\, u^j = G_{0,0}[\lambda_1, \lambda_2, \lambda_3, t] \qquad (6.37)$$

The ratio $\frac{G_{0,0}}{G_{0,0}{}'}$ can be determined in a similar manner as described above in which the ratio $\frac{g_{0,0}}{g_{0,0}{}'}$ is determined by comparing the ratio , $\frac{\Delta t'}{\Delta t}$, of observed clock intervals between a clock , t' , fixed at (x, y, z) and an identical observer's clock, t, located at the origin. $G_{0,0}{}'$ is the value of the time like quantity at the observers location in the $(\lambda_1, \lambda_2, \lambda_3)$ space. However the difference in this scenario is that the test clock, t', in the $(\lambda_1, \lambda_2, \lambda_3)$ space must be fixed at the particular location identified by the particular $(\lambda_1, \lambda_2, \lambda_3)$ with respect to the labeling space, in other words the $(\lambda_1, \lambda_2, \lambda_3)$ coordinates of the test clock , t' are fixed since $\Delta\lambda_i = 0$. So the ratio $\frac{\Delta t'}{\Delta t}$ compares the time intervals between a clock being convected along in the fluid path and the experimenter's local clock.

$$\frac{\Delta t'}{\Delta t} = \sqrt{\frac{G_{0,0}}{G_{0,0}{}'}} \qquad (6.38)$$

If we examine the particular scenario where $g_{0,i} = 0$ and $g_{i,j} = -\frac{g_{0,0}}{c^2}$, then eq (6.38) simplifies to

$$\frac{\Delta t'}{\Delta t} = \sqrt{\frac{g_{0,0}}{G_{0,0}{}'}\left(1 - \frac{V.V}{c^2}\right)} \qquad (6.39)$$

where $V = (u^i\,,\ i = 1, 2, 3)$

eq ((6.39)) combines the respective predictions of Special and General Relativity in that a moving clock ticks slower and a clock in a higher potential will tick faster.

Appendix A

Discrete case

Suppose

$$L = \sum_{i,j=1}^{n} l\,(x_i\,,\, y_i,\, z_i\,,\, x_j\,,\, y_j,\, z_j,\, u_i\,,\, v_i\,,\, w_i\,,\, u_j\,,\, v_j\,,\, w_j\,) \;=\; \sum_{i,j=1}^{n} l_{i,j} \qquad \text{(A .1)}$$

and that

$$l_{i,j} = l_{j,i} \qquad \text{(A .2)}$$

Then the Euler Lagrange equations generated by L are,

$$\frac{d}{dt}\left[\frac{\delta L_k}{\delta u_k}\right] - \frac{\delta L_k}{\delta x_k} = 0\,,\;\; \frac{d}{dt}\left[\frac{\delta L_k}{\delta v_k}\right] - \frac{\delta L_k}{\delta y_k} = 0,\;\; \frac{d}{dt}\left[\frac{\delta L_k}{\delta w_k}\right] - \frac{\delta L_k}{\delta z_k} = 0 \;\;\; k = 1,\, n \qquad \text{(A .3)}$$

where

$$L_k = \sum_{j=1}^{n} l_{k,j} \qquad j \neq k \;\text{ and }\; L = \sum_{k=1}^{n} L_k \qquad \text{(A .4)}$$

proof :

$$\frac{\delta L}{\delta u_k} = \sum_{i,j=1}^{n}\left(\frac{\delta l_{i,j}}{\delta u_i}\,\frac{\delta u_i}{\delta u_k} + \frac{\delta l_{i,j}}{\delta u_j}\,\frac{\delta u_j}{\delta u_k}\right) = \sum_{i,j=1}^{n}\left(\frac{\delta l_{i,j}}{\delta u_i}\,\delta_k{}^{i} + \frac{\delta l_{i,j}}{\delta u_j}\,\delta_k{}^{j}\right)$$

$$= \sum_{j=1}^{n}\frac{\delta l_{k,j}}{\delta u_k} + \sum_{i=1}^{n}\frac{\delta l_{i,k}}{\delta u_k} = \sum_{j=1}^{n}\frac{\delta l_{k,j}}{\delta u_k} + \sum_{j=1}^{n}\frac{\delta l_{k,j}}{\delta u_k}$$

$$= 2\sum_{j=1}^{n}\frac{\delta l_{k,j}}{\delta u_k} = 2\,\frac{\delta L_k}{\delta u_k} \qquad \text{(A .5)}$$

since $l_{k,j} = l_{j,k}$, and the substitution, $i \rightarrow j$, does not alter the meaning of the summation

Similarly it is easy to show that

$$\frac{\delta L}{\delta x_k} = 2\,\frac{\delta L_k}{\delta x_k} \tag{A .6}$$

Therefore

$$2\left(\frac{d}{dt}\left[\frac{\delta L_k}{\delta u_k}\right] - \frac{\delta L_k}{\delta x_k}\right) = \frac{d}{dt}\left[\frac{\delta L}{\delta u_k}\right] - \frac{\delta L}{\delta x_k} = 0 \tag{A .7}$$

i.e

$$\frac{d}{dt}\left[\frac{\delta L_k}{\delta u_k}\right] - \frac{\delta L_k}{\delta x_k} = 0$$

the proofs for

$$\frac{d}{dt}\left[\frac{\delta L_k}{\delta v_k}\right] - \frac{\delta L_k}{\delta y_k} = 0$$

and $$\frac{d}{dt}\left[\frac{\delta L_k}{\delta w_k}\right] - \frac{\delta L_k}{\delta z_k} = 0 \tag{A .8}$$

are obtained in the same way

Continuous case (1 dimensional)

Suppose

$$L = \int_{\lambda=0}^{1}\int_{\beta=0}^{1} l_{\lambda,\beta}\, d\lambda\, d\beta \tag{A .9}$$

such that

$$l_{\lambda,\beta} = l_{\beta,\lambda}$$

where

$$l_{\lambda,\beta} = \left(X[\lambda, t], \frac{\delta X[\lambda, t]}{\delta t}, X[\beta, t], \frac{\delta X[\beta, t]}{\delta t},\right) \tag{A .10}$$

and $X[\lambda, t]$ describes the path of a fluid in one dimension,

Then the Euler − Lagrange Equations generated by the continuous version of L are; (A .11)

$$\frac{d}{dt}\left[\frac{\delta L_\lambda}{\delta \dot{x}_\lambda}\right] - \frac{\delta L_\lambda}{\delta x_\lambda} =$$

0 \ \ \ (A .12)

where

$$L_\lambda = \int_{\lambda=0}^{1} l_{\lambda,\beta}\, d\beta \quad \text{and} \quad L = \int_{\lambda=0}^{1} L_\lambda\, d\lambda \qquad \text{(A .13)}$$

proof :

Define the Action , A, as

$$A = \int_0^T L\, dt = \int_0^T \int_{\lambda=0}^{1} \int_{\beta=0}^{1} l_{\lambda,\beta}\, d\lambda\, d\beta\, dt \qquad \text{(A .14)}$$

The equations of motion are defined such that the action is a minimum ie

s.t

$\delta A = 0$

.Therefore

$$\begin{aligned}
0 = &\int_0^T \int_{\lambda=0}^{1} \int_{\beta=0}^{1} \delta l_{\lambda,\beta}\, d\lambda\, d\beta\, dt = \int_0^T \int_{\lambda=0}^{1} \int_{\beta=0}^{1} \left(\frac{\delta l_{\lambda,\beta}}{\delta x_\lambda} \delta x_\lambda + \frac{\delta l_{\lambda,\beta}}{\delta \dot{x}_\lambda} \delta \dot{x}_\lambda + \frac{\delta l_{\lambda,\beta}}{\delta x_\beta} \delta x_\beta + \frac{\delta l_{\lambda,\beta}}{\delta \dot{x}_\beta} \delta \dot{x}_\beta \right) d\lambda\, d\beta\, dt = \\
&\int_0^T \int_{\lambda=0}^{1} \int_{\beta=0}^{1} \left(\frac{\delta l_{\lambda,\beta}}{\delta x_\lambda} \delta x_\lambda + \frac{\delta l_{\lambda,\beta}}{\delta \dot{x}_\lambda} \delta \dot{x}_\lambda \right) d\lambda\, d\beta\, dt + \int_0^T \int_{\lambda=0}^{1} \int_{\beta=0}^{1} \left(\frac{\delta l_{\lambda,\beta}}{\delta x_\beta} \delta x_\beta + \frac{\delta l_{\lambda,\beta}}{\delta \dot{x}_\beta} \delta \dot{x}_\beta \right) d\lambda\, d\beta\, dt = \\
&\int_0^T \int_{\lambda=0}^{1} \int_{\beta=0}^{1} \left(\frac{\delta l_{\lambda,\beta}}{\delta x_\lambda} \delta x_\lambda + \frac{\delta l_{\lambda,\beta}}{\delta \dot{x}_\lambda} \delta \dot{x}_\lambda \right) d\lambda\, d\beta\, dt + \int_0^T \int_{\beta=0}^{1} \int_{\lambda=0}^{1} \left(\frac{\delta l_{\lambda,\beta}}{\delta x_\lambda} \delta x_\lambda + \frac{\delta l_{\lambda,\beta}}{\delta \dot{x}_\lambda} \delta \dot{x}_\lambda \right) d\beta\, d\lambda\, dt = \\
&2 \int_0^T \int_{\lambda=0}^{1} \int_{\beta=0}^{1} \left(\frac{\delta l_{\lambda,\beta}}{\delta x_\lambda} \delta x_\lambda + \frac{\delta l_{\lambda,\beta}}{\delta \dot{x}_\lambda} \delta \dot{x}_\lambda \right) d\lambda\, d\beta\, dt \qquad \text{(A .15)}
\end{aligned}$$

(by interchanging the dummy variables , λ and β, in the second integral and using the fact that $l_{\lambda,\beta} = l_{\beta,\lambda}$)

where

$\delta x_\lambda = \delta x[\lambda, t]$

$\delta x_\beta = \delta x[\beta, t]$

$$\delta \dot{x}_\lambda = \delta \left(\frac{\delta x[\lambda, t]}{\delta t} \right) = \frac{\delta}{\delta t} \left(\delta x[\lambda, t] \right) = (\delta \dot{x}_\lambda)$$

$$\delta \dot{x}_\beta = \delta \left(\frac{\delta x[\beta, t]}{\delta t} \right) = \frac{\delta}{\delta t} \left(\delta x[\beta, t] \right) = (\delta \dot{x}_\beta)$$

Therefore

$$\delta A = 0 \ = \ 2 \int_0^T \int_{\lambda=0}^1 \int_{\beta=0}^1 \left(\frac{\delta l_{\lambda,\beta}}{\delta x_\lambda} \delta x_\lambda \ + \frac{\delta l_{\lambda,\beta}}{\delta \dot{x_\lambda}} \delta \dot{x_\lambda} \right) d\lambda\, d\beta\ dt \ =$$

$$2 \int_0^T \int_{\lambda=0}^1 \left(\frac{\delta L_\lambda}{\delta x_\lambda} \delta x_\lambda \ + \frac{\delta L_\lambda}{\delta \dot{x_\lambda}} \delta \dot{x_\lambda} \right) d\lambda\ dt \ = 2\,\delta \left(\int_0^T \int_{\lambda=0}^1 L_\lambda\, d\lambda\ dt \right) \qquad (A.16)$$

i.e

$$0 = \delta \left(\int_0^T \int_{\lambda=0}^1 L_\lambda\, d\lambda\ dt \right) \qquad (A.17)$$

which implies the Euler – Lagrange Equations;

$$\frac{d}{dt}\left[\frac{\delta L_\lambda}{\delta \dot{x_\lambda}}\right] - \frac{\delta L_\lambda}{\delta x_\lambda} = 0 \qquad (A.18)$$

Continuous case (3 dimensional)

The above analysis can be extended along the same lines in
a straightforward way to the case where the fluid path is three dimensional

i.e. if

$$L = \int_{\lambda_1=0}^1 \int_{\lambda_2=0}^1 \int_{\lambda_3=0}^1 \int_{\beta_1=0}^1 \int_{\beta_2=0}^1 \int_{\beta_3=0}^1 l \left(X[\lambda_1, \lambda_2, \lambda_3, t], \frac{\delta X[\lambda_1, \lambda_2, \lambda_3, t]}{\delta t}, X[\beta_1, \beta_2, \beta_3, t], \frac{\delta X[\beta_1, \beta_2, \beta_3, t]}{\delta t}, \right.$$
$$Y[\lambda_1, \lambda_2, \lambda_3, t], \frac{\delta Y[\lambda_1, \lambda_2, \lambda_3, t]}{\delta t}, Y[\beta_1, \beta_2, \beta_3, t], \frac{\delta Y[\beta_1, \beta_2, \beta_3, t]}{\delta t},$$
$$\left. Z[\lambda_1, \lambda_2, \lambda_3, t], \frac{\delta Z[\lambda_1, \lambda_2, \lambda_3, t]}{\delta t}, Z[\beta_1, \beta_2, \beta_3, t], \frac{\delta Z[\beta_1, \beta_2, \beta_3, t]}{\delta t} \right)$$
$$d\lambda_1\, d\lambda_2\, d\lambda_3\, d\beta_1\, d\beta_2\, d\beta_3 \qquad (A.19)$$

then

$$\frac{d}{dt}\left[\frac{\delta L_\lambda}{\delta \dot{x_\lambda}}\right] - \frac{\delta L_\lambda}{\delta x_\lambda} = 0$$
$$\frac{d}{dt}\left[\frac{\delta L_\lambda}{\delta \dot{y_\lambda}}\right] - \frac{\delta L_\lambda}{\delta y_\lambda} = 0$$
$$\frac{d}{dt}\left[\frac{\delta L_\lambda}{\delta \dot{z_\lambda}}\right] - \frac{\delta L_\lambda}{\delta z_\lambda} = 0 \qquad (A.20)$$

where in this case

$x_\lambda = X[\lambda_1, \lambda_2, \lambda_3, t]$

$$\dot{x_\lambda} = \frac{\delta X[\lambda_1, \lambda_2, \lambda_3, t]}{\delta t}$$

$y_\lambda = Y[\lambda_1, \lambda_2, \lambda_3, t]$

$$\dot{y_\lambda} = \frac{\delta Y[\lambda_1, \lambda_2, \lambda_3, t]}{\delta t}$$

$z_\lambda = Z[\lambda_1, \lambda_2, \lambda_3, t]$

$$\dot{z_\lambda} = \frac{\delta Z[\lambda_1, \lambda_2, \lambda_3, t]}{\delta t}$$

$$\begin{aligned} l\Bigg(&X[\lambda_1, \lambda_2, \lambda_3, t], \frac{\delta X[\lambda_1, \lambda_2, \lambda_3, t]}{\delta t}, X[\beta_1, \beta_2, \beta_3, t], \frac{\delta X[\beta_1, \beta_2, \beta_3, t]}{\delta t}, \\ &Y[\lambda_1, \lambda_2, \lambda_3, t], \frac{\delta Y[\lambda_1, \lambda_2, \lambda_3, t]}{\delta t}, Y[\beta_1, \beta_2, \beta_3, t], \frac{\delta Y[\beta_1, \beta_2, \beta_3, t]}{\delta t}, \\ &Z[\lambda_1, \lambda_2, \lambda_3, t], \frac{\delta Z[\lambda_1, \lambda_2, \lambda_3, t]}{\delta t}, Z[\beta_1, \beta_2, \beta_3, t], \frac{\delta Z[\beta_1, \beta_2, \beta_3, t]}{\delta t}\Bigg) = l_{\lambda,\beta}\end{aligned}$$

$$L_\lambda = \int_{\beta_1=0}^{1}\int_{\beta_2=0}^{1}\int_{\beta_3=0}^{1} l_{\lambda,\beta}\, d\beta_1\, d\beta_2\, d\beta_3 \quad = \int_{\beta_1=0}^{1}\int_{\beta_2=0}^{1}\int_{\beta_3=0}^{1} l_{\lambda,\beta}\ \rho\, dX\, dY\, dZ$$

$$\rho = \begin{pmatrix} \frac{\delta\beta_1}{\delta X} & \frac{\delta\beta_1}{\delta Y} & \frac{\delta\beta_1}{\delta Z} \\ \frac{\delta\beta_2}{\delta X} & \frac{\delta\beta_2}{\delta Y} & \frac{\delta\beta_2}{\delta Z} \\ \frac{\delta\beta_3}{\delta X} & \frac{\delta\beta_3}{\delta Y} & \frac{\delta\beta_3}{\delta Z} \end{pmatrix}$$

Null