Observations on the Equivalence of the Harmonic Gauge Constraint and Global Energy-Momentum Conservation in General Relativity

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Abstract. It is here demonstrated that by constraining the Einstein Field Equations (EFE) with the Harmonic Gauge condition results in not only the well established implication that the EFE are nothing more than wave equations, but also in the fact of the existence of a global conservation of energy and momentum law comprised of both matter and fields, with the fields being defined in terms of the underlying metric. This result is well known to practitioners of General Relativity, and it is the purpose of this paper to show rather explicitly how it is done, especially for the benefit of beginners to the field in as simple a manner as the machinery of Tensor Calculus will permit. It is also speculated, but not elaborated upon here, that the various pseudo tensor approaches to construct global conservation equations are equivalent to the EFE with a corresponding attached gauge constraint, but which may not necessarily in each case correspond to a wave like equation, unlike the case for the Harmonic Gauge

1. Notation and Conventions

Before proceeding further it will be necessary to clarify some definitions, conventions and properties, regarding the raising and lowering of tensor indices, to be employed in subsequent sections

Firstly for an arbitrary tensor, T, the index i raised with respect to the metric g, is denoted as

$$T^{i(g)} = g^{i,j}T_i \qquad (1.1)$$

and similarly for the lower index;

$$T_{i(g)} = g_{i,j}T^i \qquad (1.2)$$

It is also useful to observe the linearity of the raising and lowering operator, i.e. for arbitrary tensors $T, g1, g2$, and constants a , b

$$T^{i(a\,g1+b\,g2)} = \left(ag1^{i,j} + bg2^{i,j}\right)T_j = ag1^{i,j}T_j + bg2^{i,j}T_j = aT^{i(g1)} + bT^{i(g2)} \quad (1.3)$$

and similarly

$$T_{i(a\,g1+b\,g2)} = aT_{i(g1)} + bT_{i(g2)} \qquad (1.4)$$

For an arbitrary function f the comma operator $f_{,i}$ is defined as

$$f_{,i} = \frac{\partial f}{\partial x_i} \qquad (1.5)$$

2. The Harmonic Gauge

Suffice to say that the Harmonic Gauge when used in conjunction with EFE has the intended effect of reducing the EFE to a set of wave equations. For a fuller exposition see [2]. For the purpose of this analysis the Harmonic Gauge will be expressed in the following form:

$$(g^{i,j}\sqrt{-G})_{,i} = 0 \qquad\qquad (2.1)$$

where G represents the determinant of the metric tensor.

Observing that;

$$(g^{i,j}\sqrt{-G})_{,j} = \sqrt{-G}\; g^{i,j}{}_{,j} + g^{i,j}(\sqrt{-G})_{,j} = \sqrt{-G}\; g^{i,j}{}_{,j} + \frac{1}{2}g^{i,j}\frac{1}{\sqrt{-G}}\frac{\partial(-G)}{\partial g_{m,n}}(g_{m,n})_j$$

and

$$\frac{\partial(-G)}{\partial g_{m,n}} = -g^{n,m}\,G$$

implies that (2.1) becomes

$$(g^{i,j}\sqrt{-G})_{,j} = \sqrt{-G}\; g^{i,j}{}_{,j} + \tfrac{1}{2}g^{i,j}\sqrt{-G}\,g^{n,m}\,(g_{m,n})_{,j} = \sqrt{-G}\left(g^{i,j}{}_{,j} + \tfrac{1}{2}g^{i,j}g^{n,m}\,\left(g_{m,n}\right)_{,j}\right) = 0.$$

which in turn implies that

$$g^{i,j}{}_{,j} + \frac{1}{2}g^{i,j}g^{n,m}\left(g_{m,n}\right)_{,j} = 0 \qquad\qquad (2.2)$$

Multiplying both sides of Eq (2.2) by $g_{i,k}$ and contracting with respect to i, gives

$$g_{i,k}g^{i,j}{}_{,j} + \frac{1}{2}g_{i,k}\,g^{i,j}g^{n,m}\left(g_{m,n}\right)_{,j} = 0,$$

which after substituting the relation;

$$g_{i,k}g^{i,j}{}_{,j} = -g^{i,j}(g_{i,k})_{,j}$$

yields

$$g^{i,j}\left(g_{i,k}\right)_{,j} - \frac{1}{2}\delta_k^j g^{n,m}\left(g_{m,n}\right)_{,j} = 0 \qquad (2.3).$$

For the remainder of the analysis, Eq (2.3) will provide a more convenient representation of the HG. Let

$$g^{i,j} = \epsilon^{i,j} + \Delta(g^{i,j})$$

$$g_{i,j} = \epsilon_{i,j} + \Delta g_{i,j}$$

, in which ϵ represents the flat constant Minkowski metric, i.e.

$$\epsilon = \begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and define

$$\Delta g_k^j = \epsilon^{i,j}\Delta g_{i,k} \qquad (d.1)$$

$$\psi_k^j = \Delta g_k^j - \delta_k^j g \qquad (d.2)$$

$$g = \frac{1}{2}\Delta g_n^n \qquad (d.3)$$

$$H_k = \Delta(g^{i,j})\left(g_{i,k}\right)_{,j} - \frac{1}{2}\delta_k^j \Delta(g^{n,m})\left(g_{m,n}\right)_{,j} \qquad (d.4)$$

then it is easy to show that Eq (2.3) is equivalent to;

$$\psi_{j\ ,i}^{i} + H_j = 0 \qquad (2.4)$$

For proof of Eq (2.4) see Appendix(A). It’s important to emphasize that Eq (2.4) is a complete statement of the Harmonic Gauge ,achieved as a result of splitting the metric between a flat space-time and a departure from the Minkowski metric, and is not simply a linear approximation to Eq (2.3)

3. Einstein Field Equations

$$E_j^i = \frac{8\pi\boldsymbol{G}}{c^4} T_{m\,j}^{\ \ i} \qquad (3.1)$$

where E_j^i is the Einstein Tensor and $T_{m\,j}^{\ \ i}$ is the energy momentum tensor. For convenience and clarity in the remainder of this analysis, the coupling constant will be absorbed into the energy momentum tensor to give;

$$E_j^i = -\frac{1}{2}T_j^i$$

by employing the substitution;

$$\frac{8\pi \boldsymbol{G}}{c^4} {T_m}_j^i = -\frac{1}{2} T_j^i$$

The EFE can be recast in terms of the definitions given by Eq (d.1)-Eq (d.4) as;

$$\Omega\psi_j^i = -T_j^i - S_j^i \qquad (3.2)$$

where S_j^i represents a field source given in terms of $\Delta g_{i,j}$ and its derivatives, and the Ω wave operator is defined as;

$$\Omega = \sum_{i=1}^{3} \frac{\partial^2}{\partial {x_i}^2} - \frac{1}{c^2}\frac{\partial^2}{\partial {x_0}^2} = -\Omega_{\mathrm{i}}\Omega^{\mathrm{i}(\epsilon)}$$

$$\Omega_{\mathrm{i}} = \frac{\partial}{\partial x_i}$$

For proof of Eq (3.2) and the definition of S_j^i, see Appendix (B)

4. Energy-Momentum Conservation

Operating on Eq (2.4) with Ω gives;

$$\Omega(\psi_{j\ ,i}^{i}) + \Omega H_j = \Omega\psi_{j\ ,i}^{i} + \Omega H_j = 0 \quad (4.1).$$

From Eq (3.2) it is clear that;

$$\Omega\psi_{j\ ,i}^{i} = -(T_j^i + S_j^i)_{,i} \qquad (4.2).$$

Since

$$\Omega H_j = -(\Omega_{\mathrm{i}}\Omega^{\mathrm{i}(\epsilon)})H_j = -\Omega_{\mathrm{i}}\left(\Omega^{\mathrm{i}(\epsilon)}H_j\right) = -\Omega_{\mathrm{i}}\left(\Omega_{\mathrm{k}}\,\epsilon^{k,i}H_j\right) = -(\,\epsilon^{k,i}H_{j,k})_{,\mathrm{i}} = -({H_j}^{,\mathrm{i}(\epsilon)})_{,\mathrm{i}}$$

then Eq (4.1) and Eq (4.2) can be combined to yield;

$$(T_j^i + S_j^i + \Omega^{\mathrm{i}(\epsilon)}H_j)_{,i} = (T_j^i + S_j^i + {H_j}^{,\mathrm{i}(\epsilon)})_{,i} = 0\,.$$

Suppose for a particular three dimensional region V bounded by a surface W on which $(T_j^i + S_j^i +$ $Hj{,}\mathrm{i}\epsilon{=}0\ \ forhhhh\ i{\in}1{,}2{,}3$, and defining the energy-momentum four vector, pjas

$$p_j = T_j^0 + S_j^0 + {H_j}^{,0(\epsilon)},$$

then it would necessarily be the case that

$$\frac{\partial \iiint_V p_j \, dx_1 dx_2 dx_3}{\partial t} = 0 \qquad (4.3).$$

It's important to mention how the p_j are split between the matter components of energy-momentum T_j^0 and the field components, $S_j^0 + H_j{}'^{0(\epsilon)}$, so that obviously Eq (4.3) allows for an exchange between the total matter contributions and the total field contributions.

5. The Harmonic Gauge and the Lorentz Transformation

In what follows it will be convenient to use the following form of the Harmonic Gauge.

$$(g^{i,j}\sqrt{-G})_{,i} = 0 \qquad (5.1)$$

Of special consideration is the observation that Eq (5.1) in its present form is not a tensor equation, that is for an arbitrary transformation, $T: x_i \rightarrow x_i{}'$, it will generally be the case that;

$$(g^{i,j\prime}\sqrt{-G'})_{,i} \neq 0 \qquad (5.2)$$

However this is exactly what is required as the purpose of Eq (5.1) is to select a frame of reference. It is only under a certain class of transformations will it be the case that Eq (5.1) remains invariant, as will be shown as follows. Let us write

$$g_{i,j} = \epsilon_{i,j} + \Delta g_{i,j} \quad (5.3)$$

i.e., all we are doing is splitting the metric tensor into the sum of a constant metric, $\epsilon_{i,j}$, and a variable component, $\Delta g_{i,j}$.

Context Dependent Covariant Derivative. The context dependent covariant derivative simply makes explicit the underlying metric tensor which the covariant derivative of an arbitrary tensor is computed in the context of, i.e. for an arbitrary tensor, T^i, the covariant derivative, $T^i{}_{;j(h)}$, of T^i with respect to x_j, simply indicates that it is the underlying metric $h_{i,j}$ which is used as the background metric for the computation of the derivative. Adopting this convention means that since the $\epsilon_{i,j}$ are constant then it is possible to express Eq (5.1) as;

$$(g^{i,j}\sqrt{-G})_{;i(\epsilon)} = 0 \qquad (5.4).$$

Eq (5.4) now express's a covariant form since under any arbitrary transformation, $T: x_i \rightarrow x_i{}'$, such that $g_{i,j} \rightarrow g_{i,j}{}'$ and $\epsilon_{i,j} \rightarrow \epsilon_{i,j}{}'$ we have;

$$(g_{i,j}{}'\sqrt{-G'})_{;i(\epsilon')} = 0 \qquad (5.5)$$

and in which the $\epsilon_{i,j}{}'$ need not necessarily be constant. If we consider the special class of transformations $Lortz: x_i \rightarrow x_i{}'$ such that the metric tensor $\epsilon_{i,j}$ is mapped unto itself, i.e.

$$\epsilon = \epsilon' = Lortz[\epsilon] \quad (5.6)$$

and for completeness;

$$g' = Lortz[g] \qquad (5.7)$$

then Eq (5.5) becomes;

$$0 = (g_{i,j}{}'\sqrt{-G'})_{;i(\epsilon')} = (g_{i,j}{}'\sqrt{-G'})_{;i(\epsilon)} = (g_{i,j}{}'\sqrt{-G'})_{,i} \qquad (5.8).$$

Simply put, if $g_{i,j}$ constitutes a solution to Eq (5.1) then under the transformation given by Eq (5.7), $g_{i,j}{}'$ is also a solution, so in this sense the class of transformations, $Lortz$, can be regarded as a device for generating solutions to Eq(5.1).

Historically the transformation class represented by $Lortz$ became known as the Lorentz Transformations and forms the axiomatic basis of Special Relativity. It's also worthwhile to note that Eq (5.8) is true for the more general class of linear transformations including a Galilean transformation. This can be easily seen from the fact that under a linear transformation, $\epsilon_{i,j} \to \epsilon_{i,j}{}'$, if the $\epsilon_{i,j}$are constant then so will the $\epsilon_{i,j}{}'$.

6. The Kerr and Schartzichild Metrics in Relation to the Harmonic Gauge.

It will be briefly stated here the observation that both the Kerr and Schartzichild metrics are both inconsistent with the Harmonic Gauge, and therefore call into question their viability as physical solutions to the Einstein Field Equations as pertaining to descriptions of black holes. In other words the reality of both black holes and gravitational waves, and hence global energy-momentum conservation, seem to be at odds and cannot both exist simultaneously. To elaborate if the reality of Kerr and Schartzichild metrics are accepted as physical realities then they must correspond to gauges other than the Harmonic Gauge in which case then, gravitational waves ,or equivalently global energy-momentum conservation, can't exist and vice versa. This inconsistency can be easily demonstrated with the aid of the context dependent covariant derivative. For a spherically symmetric body of mass M, the Kerr Metric in the spherical space time coordinates, (t, r, θ, φ), is given by;

$$g_{i,j}{}' = \begin{pmatrix} (1-\frac{r_s}{r})c^2 & 0 & 0 & 0 \\ 0 & -(1-\frac{r_s}{r})^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin\theta^2 \end{pmatrix}$$

in which $r_s = \frac{2\mathbf{G}M}{c^2}$ $with\ \mathbf{G}\ inicating\ the\ gravitational\ constant$, denotes the Schartzichild radius. The flat Minkowski metric ,$\epsilon_{i,j}{}'$, in the same coordinate system is given by;

$$\epsilon_{i,j}{}' = \begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin\theta^2 \end{pmatrix}.$$

Aided with the machinery of covariant differentiation carried out with respect to $\epsilon_{i,j}{}'$ as the background metric, it is straight forward to show that it will generally be that case that;

$$(g_{i,j}{}'\sqrt{-G'})_{;i(\epsilon')} \neq 0$$

which states that in spherical polar coordinated the harmonic gauge is not satisfied. The case for the Kerr metric can be analysed in the same straightforward fashion, leading to the same conclusion. In fact with this approach it is easy to show that all the vacuum solutions to the field equations fail to satisfy the Harmonic Gauge.

Conclusion

In view of the fact of the already established variety of pseudo tensor constructions which yield a global energy momentum conservation, what is however interesting is that it would appear the Harmonic Gauge, which is required in order to infer the existence of gravitational waves, has such a law already built in. However as has been shown with the aid of the device known as the context dependent covariant derivative, the coexistence of both the vacuum solutions and gravitational waves, and by implication global energy momentum conservation, is not permitted and should inspire the exercise of caution in the treatment of such phenomena when used in conjunction with one another.

As has been mentioned in the abstract, it is conjectured that a pseudo tensor construction yielding a global conservation law actually corresponds to a as yet unknown gauge condition which is required in order to remove the four degrees of ambiguity from the EFE. As the pseudo tensor or the corresponding gauge condition actually chooses the frame of reference, and hence sacrifices covariance, then in this regard it has to be concluded that attempts to find a globally covariant law of conservation are exercises in futility, since conservation laws can only be true in those frames of reference determined by the gauge. However the stated connection between gauges and pseudo tensors is still only a conjecture and establishing a proof will form the body of a future research initiative.

Appendix A

Since

$$\begin{aligned} g^{i,j}\left(g_{i,k}\right)_{,j}-\frac{1}{2}\delta_k^j g^{n,m}\left(g_{m,n}\right)_{,j} &= (\epsilon^{i,j}+\Delta(g^{i,j}))\left(g_{i,k}\right)_{,j}-\frac{1}{2}\delta_k^j(\epsilon^{n,m}+\Delta(g^{n,m}))\left(g_{m,n}\right)_{,j} \\ &= \Delta g^j_{k,j}-\delta_k^j g_{,j}+\Delta(g^{i,j})\left(g_{i,k}\right)_{,j}-\frac{1}{2}\delta_k^j\Delta(g^{n,m})\left(g_{m,n}\right)_{,j} = \Delta g^j_{k,j}-\delta_k^j g_{,j}+H_k \\ &= \Delta g^j_{k,j}-\delta_k^j g_{,j}+H_k = (\Delta g^j_k-\delta_k^j g)_{,j}+H_k = \psi^i_{k,i}+H_k = 0 \end{aligned}$$

in which

$$H_k = \Delta(g^{i,j})\left(g_{i,k}\right)_{,j}-\frac{1}{2}\delta_k^j\Delta(g^{n,m})\left(g_{m,n}\right)_{,j}$$

Appendix B

In order to arrive at Eq (3.2) we begin by splitting the metric in the definition of the Cristoffel Symbols as follows;

$$\begin{aligned}
\Gamma^m_{i,j} &= \frac{1}{2} g^{k,m}\left(\left(g_{i,k}\right)_{,j} + \left(g_{j,k}\right)_{,i} - \left(g_{i,j}\right)_{,k}\right) \\
&= \frac{1}{2}\left(\epsilon^{k,m} + \Delta(g^{k,m})\right)\left(\left(\Delta g_{i,k}\right)_{,j} + \left(\Delta g_{j,k}\right)_{,i} - \left(\Delta g_{i,j}\right)_{,k}\right) \\
&= \frac{1}{2}\left(\Delta g^{m}_{i\ ,j} + \Delta g^{m}_{j\ ,i} - \Delta g_{\ i,j}^{\ \ ,m}\right) + \frac{1}{2}\Delta(g^{k,m})\left(\left(\Delta g_{i,k}\right)_{,j} + \left(\Delta g_{j,k}\right)_{,i} - \left(\Delta g_{i,j}\right)_{,k}\right) \\
&= {\Gamma_g}^{m}_{i,j} + {\Gamma_{gg}}^{m}_{i,j} \qquad (B.1)
\end{aligned}$$

in which the linear and quadratic terms are respectively given by;

$${\Gamma_g}^{m}_{i,j} = \frac{1}{2}\left(\Delta g^{m}_{i\ ,j} + \Delta g^{m}_{j\ ,i} - \Delta g_{\ i,j}^{\ \ ,m}\right)$$

and

$${\Gamma_{gg}}^{m}_{i,j} = \frac{1}{2}\Delta(g^{k,m})\left(\left(\Delta g_{i,k}\right)_{,j} + \left(\Delta g_{j,k}\right)_{,i} - \left(\Delta g_{i,j}\right)_{,k}\right).$$

The Curvature Tensor then becomes;

$$\begin{aligned}
R^i_{j,k,l} &= {\Gamma^i_{j,l}}_{,k} - {\Gamma^i_{j,k}}_{,l} + \Gamma^m_{j,l}\Gamma^i_{m,k} - \Gamma^m_{j,k}\Gamma^i_{m,l} \\
&= \left({{\Gamma_g}^{i}_{j,l}}_{,k} + {{\Gamma_{gg}}^{i}_{j,l}}_{,k}\right) - \left({{\Gamma_g}^{i}_{j,k}}_{,l} + {{\Gamma_{gg}}^{i}_{j,k}}_{,l}\right) + \Gamma^m_{j,l}\Gamma^i_{m,k} - \Gamma^m_{j,k}\Gamma^i_{m,l} \\
&= \left({{\Gamma_g}^{i}_{j,l}}_{,k} - {{\Gamma_g}^{i}_{j,k}}_{,l}\right) + \left({{\Gamma_{gg}}^{i}_{j,l}}_{,k} - {{\Gamma_{gg}}^{i}_{j,k}}_{,l}\right) + \Gamma^m_{j,l}\Gamma^i_{m,k} - \Gamma^m_{j,k}\Gamma^i_{m,l} \\
&= {R_g}^{i}_{j,k,l} + {R_{gg}}^{i}_{j,k,l} \qquad (B.2)
\end{aligned}$$

where

$${R_g}^{i}_{j,k,l} = \left({{\Gamma_g}^{i}_{j,l}}_{,k} - {{\Gamma_g}^{i}_{j,k}}_{,l}\right)$$

and

$${R_{gg}}^{i}_{j,k,l} = \left({{\Gamma_{gg}}^{i}_{j,l}}_{,k} - {{\Gamma_{gg}}^{i}_{j,k}}_{,l}\right) + \Gamma^m_{j,l}\Gamma^i_{m,k} - \Gamma^m_{j,k}\Gamma^i_{m,l}\,.$$

Contracting with respect to i and k gives;

$$R_{i,j} = {R_g}_{i,j} + {R_{gg}}_{i,j}\,.$$

After employing the index raising methodology as described in the Notation and Conventions section, we have

$R^{i(g)}_{j} = R^{i(\epsilon)}_{j} + R^{i(\Delta g)}_{j} = {R_g}^{i(\epsilon)}_{j} + {R_{gg}}^{i(\epsilon)}_{j} + R^{i(\Delta g)}_{j}$,

(as the matrix $g = \epsilon + \Delta g$)

which after substituting in the EFE finally yields

$$\begin{aligned} E_j^i = E_j^{i(g)} = R_j^{i(g)} - \frac{1}{2}\delta_j^i R_n^{n(g)} = R_{g_j}{}^{i(\epsilon)} + R_{gg_j}{}^{i(\epsilon)} + R_j^{i(\Delta g)} \\ -\frac{1}{2}\left(R_{g_n}{}^{n(\epsilon)} + R_{gg_n}{}^{n(\epsilon)} + R_n^{n(\Delta g)}\right)\delta_j^i \\ = R_{g_j}{}^{i(\epsilon)} - \frac{1}{2}R_{g_n}{}^{n(\epsilon)}\delta_j^i + R_{gg_j}{}^{i(\epsilon)} + R_j^{i(\Delta g)} - \frac{1}{2}\left(R_{gg_n}{}^{n(\epsilon)} + R_n^{n(\Delta g)}\right)\delta_j^i \\ = -\frac{1}{2}T_j^i \qquad (B.3). \end{aligned}$$

It is straightforward to show that;

$$R_{g_{i,j}} = \Gamma_{g}{}^{i}_{j,l_{,k}} - \Gamma_{g}{}^{i}_{j,k_{,l}} = \frac{1}{2}\left(\Omega\Delta g_{\ i,j} + \Delta g^n_{i\ ,n,j} + \Delta g^n_{j\ ,n,i} - 2\boldsymbol{g}_{,i,j}\right) \qquad (B.4).$$

Now since

$$\begin{aligned} g^{i,j}(g_{i,k})_{,j} - \frac{1}{2}\delta_k^j g^{n,m}(g_{m,n})_{,j} = \Delta g^j_{k_{,j}} - \delta_k^j g_{,j} + \Delta(g^{i,j})(g_{i,k})_{,j} - \frac{1}{2}\delta_k^j\Delta(g^{n,m})(g_{m,n})_{,j} \\ = \Delta g^j_{k_{,j}} - \delta_k^j\boldsymbol{g}_{,j} + H_k = \Delta g^j_{k_{,j}} - \boldsymbol{g}_{,k} + H_k = 0 \end{aligned}$$

with H_k being given by

$$H_k = \Delta(g^{i,j})(g_{i,k})_{,j} - \frac{1}{2}\delta_k^j\Delta(g^{n,m})(g_{m,n})_{,j} \quad ,$$

and also noting that consequently

$$\begin{aligned} \Delta g^n_{i\ ,n,j} = \boldsymbol{g}_{,i,j} + H_{i_{,j}} \\ \Delta g^n_{j\ ,n,i} = \boldsymbol{g}_{,j,i} + H_{j_{,i}} = \boldsymbol{g}_{,i,j} + H_{j_{,i}}, \end{aligned}$$

Eq (B.4) becomes;

$$R_{g_{i,j}} = \Gamma_{g}{}^{i}_{j,l_{,k}} - \Gamma_{g}{}^{i}_{j,k_{,l}} = \frac{1}{2}\left(\Omega\Delta g_{\ i,j} + H_{i_{,j}} + H_{j_{,i}}\right) \qquad (B.5).$$

As a result of Eq (B.5) we have

$$\mathrm{R}_{g_{\mathrm{j}}}{}^{\mathrm{i}(\epsilon)} = \frac{1}{2}(\Omega\Delta g^{\mathrm{i}}_j + H^{\mathrm{i}(\epsilon)}{}_{,j} + H_j{}^{,\mathrm{i}(\epsilon)})$$

and

$$\begin{aligned}\mathrm{R}_{g_{\mathrm{j}}}{}^{\mathrm{i}(\epsilon)}-\frac{1}{2}\mathrm{R}_{g_{\mathrm{n}}}{}^{\mathrm{n}(\epsilon)}\delta_j^i &= \frac{1}{2}\left(\Omega\Delta g_j^i+H^{\mathrm{i}(\epsilon)}{}_{,j}+H_j{}^{,\mathrm{i}(\epsilon)}-\frac{1}{2}\left(2\Omega g+H^{\mathrm{n}(\epsilon)}{}_{,n}+H_n{}^{,n(\epsilon)}\right)\delta_j^i\right)\\ &= \frac{1}{2}\left(\Omega\psi_j^i+H^{\mathrm{i}(\epsilon)}{}_{,j}+H_j{}^{,\mathrm{i}(\epsilon)}-\frac{1}{2}\left(H^{\mathrm{n}(\epsilon)}{}_{,n}+H_n{}^{,n(\epsilon)}\right)\delta_j^i\right)\\ &= \frac{1}{2}\Omega\psi_j^i+\frac{1}{2}\left(H^{\mathrm{i}(\epsilon)}{}_{,j}+H_j{}^{,\mathrm{i}(\epsilon)}-\frac{1}{2}\left(H^{\mathrm{n}(\epsilon)}{}_{,n}+H_n{}^{,n(\epsilon)}\right)\delta_j^i\right) \qquad (B.6).\end{aligned}$$

Combining Eq (B.3) and Eq (B.6) yields;

$$\begin{aligned}\frac{1}{2}\Omega\psi_j^i+\frac{1}{2}\left(H^{i(\epsilon)}{}_{,j}+H_j{}^{,i(\epsilon)}-\frac{1}{2}\left(H^{n(\epsilon)}{}_{,n}+H_n{}^{,n(\epsilon)}\right)\delta_j^i\right)+R_{gg_j}{}^{i(\epsilon)}+R_j^{i(\Delta g)}\\ -\frac{1}{2}\left(R_{gg_n}{}^{n(\epsilon)}+R_n^{n(\Delta g)}\right)\delta_j^i=-\frac{1}{2}T_j^i\end{aligned}$$

or more compactly as;

$$\Omega\psi_j^i=-T_j^i-S_j^i$$

in which, S_j^i, the field source term is given by;

$$\begin{aligned}S_j^i=H^{i(\epsilon)}{}_{,j}+H_j{}^{,i(\epsilon)}-\frac{1}{2}\left(H^{n(\epsilon)}{}_{,n}+H_n{}^{,n(\epsilon)}\right)\delta_j^i+2(R_{gg_j}{}^{i(\epsilon)}+R_j^{i(\Delta g)})\\ -\left(R_{gg_n}{}^{n(\epsilon)}+R_n^{n(\Delta g)}\right)\delta_j^i\end{aligned}$$

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